Introduction to Non-commutative Geometry

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Contents

Preface

These are the notes for a course of lectures given at the Erwin Schrödinger International Institute for Mathematics and Physics, University of Vienna in October 2022. https://www.esi.ac.at/events/e473/

The lectures were subsequently given, in a shorter form, at the Galileo Galilei Institute for Theoretical Physics in June 2023. https://www.ggi.infn.it/showevent.pl?id=446

The course is an introduction to aspects of non-commutative geometry via spectral triples, and the use of this formalism in physics. The emphasis is on introducing a number of examples that motivate and illustrate key aspects of the theory. In particular, the non-commutative geometry of the fields in the standard model was instrumental in developing the whole theory. Although some of the detail is a bit complicated, an account of spectral triples would not be complete without it.

There are exercises embedded in the text. Doing these exercises is an important part of learning the subject!

The notes are currently in draft form (and the last few lectures are not yet written). The latest version is available on my website https://johnwbarrett.wordpress.com/, and versions for the lecture courses are on the institute course pages.

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Chapter 1

Introduction

1.1 Non-commutativity

If M is a set (usually a manifold) then complex functions $f, f' \colon M \to \mathbb{C}$ can be multiplied pointwise to form a new function ff',

$$(ff')(x) = f'(x)f(x), \tag{1.1}$$

and, obviously, $ff' = f'f$.

Non-commutative geometry is about generalising the old (commutative) geometry to a situation where there are things like functions $f, f', \ldots$ which do not have to commute, so that for at least some of them,

$$ff' \neq ff'. \tag{1.2}$$

There are lots of different sorts of non-commutative geometry. They differ both in the type of geometry that is being considered and also in the methods.

One type of non-commutative geometry is already familiar to physicists: quantum mechanics. Here, functions of phase space $f(x,p)$ are replaced by their 'quantisation' $\widehat{f}$ as operators on a Hilbert space, and these typically do not commute with each other. In this generalisation of phase space geometry, some things are very similar to the classical (commutative) case. For example, the equations of motion of a harmonic oscillator are directly analogous. But other phenomena are completely new, for example the quantisation of energy levels, or the uncertainty principle, which means that there are no precise points of phase space in the quantum theory.

In these lectures I'm interested in non-commutative versions of metric geometry (Riemannian or Lorentzian manifolds, for example). These non-commutative geometries have some features in common with quantum mechanics, such as the use of matrices, or operators in Hilbert space, but non-commutative Riemannian geometry is, in general, not a quantum theory of anything. The best way to think of it is that it is a generalisation of the usual geometry.

1.2 Algebras

Before starting non-commutative geometry, it is necessary to rework the foundations of (commutative) geometry to shift the emphasis from a manifold as a set of points to viewing the functions on the manifold as the primary object.

An algebra is a vector space with a bilinear multiplication law $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$. By default it is associative, $(ab)c = a(bc)$, and unital, which means there is an identity element 1, with $1a = a = a1$. There are non-associative or non-unital algebras, but these will always be explicitly labelled (e.g., a Lie algebra is both non-associative and non-unital).

In commutative geometry, the set $\mathcal{A}$ of all continuous functions on M is not only a vector space, but it has the bilinear multiplication law $(f, f') \mapsto ff'$ that makes it a commutative algebra.

The points of M can be recovered from this algebra. A character $\theta \colon \mathcal{A} \to \mathbb{C}$ is a linear map that is a homomorphism

$$\theta(ff') = \theta(f)\theta(f'). \tag{1.3}$$

A point x determines the character

$$\theta_x(f) = f(x) \tag{1.4}$$

described as 'evaluate at me'. It turns out that all characters are of this form for some x, so the set M can be described as the set of all characters θ. Thus M can be reconstructed from the commutative algebra $\mathcal{A}$.

It is possible to go further and explain exactly what type of commutative algebras $\mathcal{A}$ correspond to topological spaces. For example, the Gelfand representation theorem says that a commutative C*-algebra corresponds to a compact topological space (see e.g., [13] for the details). The $*$ in the name C*-algebra refers to an antilinear (conjugate-linear) map $*\colon \mathcal{A} \to \mathcal{A}$ satisfying

$$a^{**} = a \quad \text{and} \quad (ab)^* = b^* a^*, \tag{1.5}$$

(which is of course equal to $a^* b^*$ for a commutative algebra). For the functions on M, the $*$ is just complex conjugation of the value of the function, $f^*(x) = (f(x))^*$. Star structures will play an important role in non-commutative geometry also.

I'm not going to explain the rest of the definition of a C*-algebra, because the detail is not going to be needed. The important point is that the study of topological spaces is equivalent to the study of certain types of commutative algebras. The set of characters becomes a topological space, rather than just a set of points, because C*-algebras have more structure than plain algebras.

The simplest example of a non-commutative geometry has $\mathcal{A} = M_N(\mathbb{C})$, the $N \times N$ matrices with complex coefficients (this is in fact a C*-algebra). On its own, this does not contain any information (beside N) so there is no geometric content to this algebra.

The concept of a point for this example is not very useful. A character $\theta \colon \mathcal{A} \to \mathbb{C}$ obeys the relation

$$\theta([a,b]) = \theta(ab - ba) = \theta(a)\theta(b) - \theta(b)\theta(a) = 0, \tag{1.6}$$

so it is non-zero only on the scalar multiples of the unit matrix I. Since $\theta(I) = 1$, there is only one 'point' in this non-commutative 'space'. In more complicated examples, there may be more than one homomorphism and so this concept of point can be useful[1].

Now choose a set of elements $X_1, X_2, \ldots, X_K \in \mathcal{A}$ that are anti-Hermitian, i.e., $X_i^* = -X_i$. One can make an operator $\Delta \colon \mathcal{A} \to \mathcal{A}$ by the formula

$$a \mapsto -[X_1,[X_1,a]] - [X_2,[X_2,a]] - \cdots - [X_K,[X_K,a]]. \tag{1.7}$$

[1] There's another concept of a point, namely, a pure state on $\mathcal{A}$ [15], defined in the same way that one defines a state in quantum mechanics. In the commutative case, these 'quantum points' are exactly the same as the characters described here, but when the algebra is non-commutative there are more quantum points than points.

This is a non-commutative analogue of the Laplace operator on a manifold M given in local coordinates by

$$\Delta_g = -g^{ij}\partial_i\partial_j + \text{lower order terms} \tag{1.8}$$

The term that is highest order (in the derivatives) is second order, and its coefficients $g^{ij}(x)$ define an inverse metric for the manifold. Thus specifying the right sort of differential operator gives the manifold a Riemannian geometry.

In a similar way, one can hypothesise that the non-commutative operator Δ determines a metric geometry for the non-commutative space $\mathcal{A}$. This is called a fuzzy space, and will be studied in more detail in Chs 2 and 3. Fuzzy spaces based on the Dirac operator, rather than the Laplacian, are studied from Ch 9 onwards.

Exercise 1. Let $\mathcal{A}$ be the algebra of diagonal $N \times N$ matrices,

$$\mathcal{A} = \left\{ \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & & \\ \vdots & & \ddots & \\ 0 & & & \lambda_N \end{pmatrix} \right\}.$$

What are the points of this space? How many points are there?

Chapter 2

Fuzzy spaces with symmetry

Fuzzy spaces were introduced briefly in Ch 1.2. These have the algebra $\mathcal{A} = M_N(\mathbb{C})$, and the geometry determined by the Laplace operator Δ in (1.7).

Particular examples of fuzzy spaces can be constructed where the algebra elements X_i are chosen to be the generators of a Lie algebra of symmetries. One can analyse the eigenvalues and eigenvectors of the Laplacian operator easily. There is an analogous commutative geometry, and one can compare with the Laplacian on this commutative geometry. It is also possible to see what the analogues of functions and vector fields are in the non-commutative case.

First, a few general points. The algebra has a star-structure a^* given by Hermitian conjugation of matrices, satisfying (1.5). There is also a Hermitian inner product on the vector space $\mathfrak{h} = M_N(\mathbb{C})$ given by

$$\langle a, b\rangle = \mathrm{Tr}(a^* b), \tag{2.1}$$

which makes $\mathfrak{h}$ a Hilbert space. This inner product is positive definite because $\langle a, a\rangle = \sum_{ij}(a^*)_{ji}a_{ij} = \sum_{ij}(a_{ij})^* a_{ij} > 0$ if $a \neq 0$.

The adjoint of an operator $[X, \cdot\,]$ is computed from

$$\begin{aligned}\langle a, [X, b]\rangle = \mathrm{Tr}\left(a^*(Xb - bX)\right) &= \mathrm{Tr}\left((a^*X - Xa^*)b\right) \\ &= \langle [X^*, a], b\rangle \end{aligned} \tag{2.2}$$

Now to the main assumption of this chapter. This is that there is a Lie group G that has a unitary action on $\mathbb{C}^N$, and the elements X in the Laplacian are the corresponding representatives of an element of its Lie algebra, $\mathrm{Lie}(G)$. Now, since X is an anti-Hermitian matrix,

$$\langle a, [X, b]\rangle = -\langle [X, a], b\rangle, \tag{2.3}$$

a rule that is analogous to integration by parts if one thinks of $[X, \cdot\,]$ as differentiation.

The identity shows that $[X, \cdot\,]$ is an anti-Hermitian operator on $\mathfrak{h}$, and $[X,[X,\cdot\,]]$ is Hermitian. Therefore Δ has real eigenvalues.

Furthermore,

$$-\langle a,[X,[X,a]]\rangle = \langle [X,a],[X,a]\rangle \geq 0, \tag{2.4}$$

so Δ, as sum of such terms, is in fact a non-negative operator on $\mathfrak{h}$.

2.1 Commutator action

The important point about the linear map $\rho(X) = [X, \cdot\,]$ is that it is a representation of $\mathrm{Lie}(G)$ on the Hilbert space $\mathfrak{h}$. To understand what this representation is in terms of conventional representation theory, the isomorphism

$$\mathfrak{h} = M_{\mathbb{N}}(\mathbb{C}) \cong \mathbb{C}^N \otimes (\mathbb{C}^N)^*, \tag{2.5}$$

is used, the second factor on the right being the dual vector space.

To check how this isomorphism works, a matrix $a \in \mathfrak{h}$ can be written as a sum of terms like $a = \psi \otimes \phi$, for $\psi \in \mathbb{C}^N$, $\phi \in (\mathbb{C}^N)^*$. This matrix multiplying a vector ξ is $a\xi = \phi(\xi)\psi$

Recall that the standard definition of a representation of $g \in G$ in $\mathbb{C}^N \otimes (\mathbb{C}^N)^*$ is

$$\psi \otimes \phi \mapsto g\psi \otimes (g^{-1})^T \phi. \tag{2.6}$$

The problem is to compute this action on a matrix, using the isomorphism (2.5).

Turning (2.6) back into a matrix by multiplying vector ξ gives

$$\big((g^{-1})^T \phi(\xi)\big) g\psi = \phi(g^{-1}\xi) g\psi = gag^{-1}\xi. \tag{2.7}$$

Therefore the action of g on the matrix a is just

$$a \mapsto gag^{-1}. \tag{2.8}$$

Writing $g = \exp X$ and linearising gives the action of X,

$$a \mapsto Xa - aX. \tag{2.9}$$

This shows that the commutator action $\rho(X) = [X, \cdot]$ is equivalent to the action of the Lie algebra on $\mathbb{C}^N$ tensored with its dual representation.

2.2 Sphere

The simplest example is the fuzzy sphere [12]. Here X_1, X_2, X_3 are the standard generators of the Lie algebra of SU(2) in the N-dimensional irreducible representation. These satisfy $[X_1, X_2] = X_3$, and cyclic permutations. Representations of SU(2) are all self-dual, since there is an invariant non-degenerate bilinear form in each representation, giving the isomorphism $\mathbb{C}^N \to (\mathbb{C}^N)^*$. Thus

$$\mathfrak{h} = M_N(\mathbb{C}) \cong \mathbb{C}^N \otimes \mathbb{C}^N \cong \mathbb{C}^1 \oplus \mathbb{C}^3 \oplus \ldots \oplus \mathbb{C}^{2N-1} \quad (2.10)$$

giving the Clebsch-Gordan decomposition into the integer-spin irreducible representations of SU(2).

The Laplacian (1.7) is

$$\Delta = -\rho(X_1)\rho(X_1) - \rho(X_2)\rho(X_2) - \rho(X_3)\rho(X_3), \quad (2.11)$$

which is the quadratic Casimir operator for the representation of G on $\mathfrak{h}$. It is called the 'total angular momentum' in physics. It has eigenvalue $j(j+1)$ on the spin j representation in (2.10), which has dimension $2j+1$. Thus the eigenvalues of Δ are

eigenvalue	0	2	6	...	$N(N-1)$
multiplicity	1	3	5	...	$2N-1$
spin	0	1	2	...	$N-1$

(2.12)

It is interesting to see what the eigenvectors look like. For example, if $N = 3$,

$$M_3(\mathbb{C}) \cong \mathbb{C}^1 \oplus \mathbb{C}^3 \oplus \mathbb{C}^5. \quad (2.13)$$

Using the weight vector basis for the original $\mathbb{C}^3$ representation (eigenvectors of X_3, say), the matrices in the $\mathbb{C}^1 \subset \mathbb{C}^3 \otimes \mathbb{C}^3$ subspace are

$$\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix},$$

the matrices in the $\mathbb{C}^3$ subspace are

$$\begin{pmatrix} b & d & 0 \\ c & 0 & d \\ 0 & c & -b \end{pmatrix},$$

while the matrices in $\mathbb{C}^5$ are the rest. It is interesting to note that the lower the eigenvalue, the closer the non-zero matrix entries are to the diagonal. This is true for general N.

Exercise 2. Modify the Laplace operator Δ for the fuzzy sphere by keeping only the first two terms, to give

$$\Delta' = -[X_1, [X_1, \cdot\,]] - [X_2, [X_2, \cdot\,]] \tag{2.14}$$

Work out a formula for the eigenvalues of Δ' by considering the eigenvalues of the difference $\Delta - \Delta'$.

Chapter 3

Commutative analogues

The aim of this chapter is to explore the commutative analogues of the fuzzy sphere and its generalisations. The discussion is with respect to the sphere, but generalisations to other examples are indicated at the end.

3.1 Spectrum

The commutative analogue of the fuzzy sphere is the ordinary sphere $S^2 \subset \mathbb{R}^3$ determined by the equation $r_1^2 + r_2^2 + r_3^2 = 1$. The functions on the sphere form a Hilbert space $\mathfrak{h}_\infty = L^2(S^2)$. The Lie group SU(2) acts on S^2 via the standard homomorphism SU(2) $\to$ SO(3), and hence on the functions, $f(r) \mapsto f(g^{-1}r)$, writing $r = (r_1, r_2, r_3)$. The representation on the Hilbert space can be decomposed into its irreducible components. This is the well-known spherical harmonic decomposition of functions on the sphere

$$\mathfrak{h}_\infty = \bigoplus_{j=0}^{\infty} \mathbb{C}^{2j+1} \tag{3.1}$$

in which every integer-spin representation appears once. This is the same as (2.10), except that there is no upper limit to the direct sum.

This decomposition means that there is a linear map that is a unitary inclusion of representations

$$\beta\colon \mathfrak{h} = M_N(\mathbb{C}) \to L^2(S^2) = \mathfrak{h}_\infty \tag{3.2}$$

It's an interesting problem to write an explicit formula that takes a matrix to a function on the sphere.

Differentiating with respect to g gives the action of the Lie algebra by the vector fields

$$V_1 = -\left(r_2 \frac{\partial}{\partial r_3} - r_3 \frac{\partial}{\partial r_2} \right), \tag{3.3}$$

and cyclic permutations of indices to give V_2 and V_3. One can check that $[V_1, V_2] = V_3$, etc.

The Laplacian on the sphere is

$$\Delta_\infty = -V_1^2 - V_2^2 - V_3^2 \tag{3.4}$$

which is also the quadratic Casimir for the SU(2) representation. Thus Δ_∞ has the same spectrum (set of eigenvalues) as Δ as shown in (2.12), except that $N \to \infty$.

3.2 Vector fields

A good question is why does one expect that the non-commutative analogue of a vector field is a commutator? The answer lies in the translation of a mapping of spaces into the algebraic framework. A homomorphism of algebras $\Phi\colon \mathcal{A}' \to \mathcal{A}$ maps a character θ of $\mathcal{A}$ to the character

$$a \mapsto \theta(\Phi(a)) \tag{3.5}$$

of $\mathcal{A}'$. So, in the commutative case, this maps points of the space M corresponding to $\mathcal{A}$ to points of the space M'.

For non-commutative algebras, the mapping of points is not a good starting point but homomorphisms of algebras (in the opposite direction) are an effective substitute. A vector field for $\mathcal{A}$ is an infinitesimal homomophism of $\mathcal{A}$ to itself (a derivation).

In the case of $\mathcal{A} = M_N(\mathbb{C})$ there is a standard result (the Skolem-Noether theorem) that any automorphism is inner, i.e.,

$$a \mapsto uau^{-1} \tag{3.6}$$

for some invertible $u \in \mathcal{A}$. The infinitesimal version, writing $u = \exp X$ and linearising, is the derivation

$$a \mapsto [X, a]. \tag{3.7}$$

To preserve the inner product on $\mathfrak{h}$, u should be chosen to be a unitary element, $u^* = u^{-1}$, and hence X is anti-Hermitian, $X^* = -X$.

Exercise 3. Show that if an inner automorphism $\phi(a) = uau^{-1}$ obeys $\langle \phi(a), \phi(b) \rangle = \langle a, b \rangle$ for all $a, b \in \mathfrak{h}$, then u^*u is a multiple of I. How can a unitary element be constructed to give the same automorphism?

For more complicated algebras, not every automorphism is inner. For example, if $\mathcal{A} = M_N(\mathbb{C}) \oplus M_N(\mathbb{C})$, there is an automorphism that interchanges the two blocks. This is not inner (it permutes the two points of this space, as in the commutative case).

3.3 Functions

The novel feature of the non-commutative case is that the Hilbert space $\mathfrak{h}$ has *two* actions of the algebra $\mathcal{A}$, namely on the left, as $a\psi$, for $a \in \mathcal{A}$ and $\psi \in \mathfrak{h}$, and on the right, ψa.

Therefore there is a left action of symmetry group G, by $g\phi$ and a right action of G, by ψg. This can be viewed as a left action of the larger group $G \times G$, by

$$(g, h) \cdot \psi = g\psi h^{-1}. \tag{3.8}$$

Similarly, the Lie algebra $\mathrm{Lie}(G) \oplus \mathrm{Lie}(G)$ acts by

$$(X, Y) \cdot \psi = X\psi - \psi Y. \tag{3.9}$$

The diagonal actions, with $g = h$ or $X = Y$, have already been used for the automorphisms and the vector fields.

What of the anti-diagonal? There's a plausible case for regarding the $X = -Y$ action as a non-commutative generalisation of functions on the 'space'. This is

$$(X, -X) \cdot \psi = X\psi + \psi X = \{X, \psi\} \tag{3.10}$$

using the notation $\{\cdot, \cdot\}$ for an anticommutator of matrices.

Applying these general ideas to the fuzzy sphere gives, for $l = 1, 2, 3$, the operators

$$V_l\psi = [X_l, \psi] \tag{3.11}$$

$$R_l\psi = \{X_l, \psi\}. \tag{3.12}$$

The first of these, V_l, is just a shorthand for the notation $\rho(X_l)$ used in Chapter 2.

These operators obey the relations

$$V_l^2 + R_l^2 = 2\{X_l^2, \cdot\} \tag{3.13}$$

$$R_l V_l = [X_l^2, \cdot\,] \tag{3.14}$$

for each fixed l.

Exercise 4. Prove (3.13) and (3.14).

Since the left or right action of $\sum_l X_l^2$ is a Casimir operator in the irreducible $\mathbb{C}^N$, it is a constant matrix

$$\sum_l X_l^2 = -\frac{N^2-1}{4}\, I_N \tag{3.15}$$

Hence summing over l gives

$$\Delta - \sum_l R_l^2 = (N^2-1) I_{\mathfrak{h}} \tag{3.16}$$

$$\sum_l R_l V_l = 0. \tag{3.17}$$

Now to compare with the commutative situation. Since β is a map of representations,

$$\beta\Delta = \Delta_\infty \beta \tag{3.18}$$

so that Δ (considered as an operator on $\mathfrak{h}_\infty$) is just a cut-off version of the commutative Laplacian. So in fact

$$\lim_{N\to\infty} \Delta = \Delta_\infty \tag{3.19}$$

as operators on $\mathfrak{h}_\infty$, using pointwise convergence in Hilbert space ($\lim_{N\to\infty} \Delta\psi = \Delta_\infty\psi$ for all $\psi \in \mathfrak{h}$).

The operators R_l need to be rescaled to converge, as is clear from (3.16). Defining the Hermitian operator $r_l = \frac{i}{N} R_l$, and dividing the equation by N^2,

$$\frac{1}{N^2}\Delta + \sum_l r_l^2 = (1 - \frac{1}{N^2}) I_{\mathfrak{h}} \tag{3.20}$$

$$\sum_l r_l V_l = 0. \tag{3.21}$$

Taking the limit $N \to \infty$ gives the equations on $\mathfrak{h}_\infty$

$$\sum_l r_l^2 = I \tag{3.22}$$

$$\sum_l r_l V_l = 0. \tag{3.23}$$

since $\frac{1}{N^2}\Delta \to 0$. These equations characterise a commutative sphere. The tangent space at a point on S^2 is two-dimensional, so there is one linear relation between three vector fields. This relation is (3.23).

Exercise 5. Calculate $[R_1, R_2]$. Explain why $[r_1, r_2] \to 0$ in the limit $N \to \infty$.

Chapter 4

Dirac operator

The Dirac operator is both more sophisticated than the Laplace operator and also more primitive, since one gets a Laplace operator by squaring a Dirac operator. It is also much more intimately connected with high-energy physics. For these reasons, the spectral triple approach to geometry is based on the Dirac operator. The Dirac operator on a manifold is explained here, with technical details that will be useful later.

4.1 Global approach

Let M be a manifold of dimension n with a metric tensor g. The cotangent bundle T^*M is the vector bundle with fibres the covectors, also called 1-forms. These covectors have a metric that is the inverse metric to g. A bundle of spinors is a complex vector bundle S with a Clifford multiplication $c\colon T^*M\otimes S \to S$ that acts pointwise. A covariant derivative acts on the sections of S, denoted $\Gamma(S)$, giving a section of $T^*M \otimes S$. The Dirac operator $D\colon \Gamma(S) \to \Gamma(S)$ is

$$D\psi = c(\nabla\psi), \tag{4.1}$$

i.e., the one-forms produced by covariant differentiation are multiplied back onto the spinors.

A number of axioms are required for this formula to work properly. It is possible to give these globally (i.e., on the whole manifold) but it is more illuminating, and leads more directly to calculations, to give these details in local coordinates in which the bundles are all trivial.

4.2 Spinors

The construction is based on a set of gamma matrices $\gamma^a \in M_k(\mathbb{C})$, $a = 1, 2, \dots n$, satisfying

$$\gamma^a\gamma^b + \gamma^b\gamma^a = -2\eta^{ab}I \tag{4.2}$$

with η^{ab} a diagonal matrix with diagonal entries ± 1. Since $(\gamma^a)^2 = -\eta^{aa}$, these entries determine whether the gamma matrices square to 1 or -1. If there are p that square to 1 and q that square to -1, the gamma matrices are called type (p, q). Clearly, $n = p + q$.

Let $V = \mathbb{C}^k$ be the vector space on which the gamma matrices act. It is possible to show that V decomposes into a sum of irreducible representations of the Clifford algebra generated by the gamma matrices. Thus the gamma matrices themselves are block diagonal on those irreducible representations. By default, it will be assumed that V is an irreducible representation.

Up to equivalence by change of basis, there is exactly one irreducible representation if n is even and two inequivalent representations if n is odd. These have dimension $k = 2^l$ for $l = n/2$ in the even n case, and $l = (n-1)/2$ for the odd case. The gamma matrices that square to 1 can be chosen Hermitian and the ones squaring to -1 anti-Hermitian, both with respect to the standard inner product

$$\langle \psi, \phi \rangle = \sum_1^k \psi_i^* \phi_i. \tag{4.3}$$

(For more detail on all of these points, see [3].)

The chirality operator is an important structure that comes for free on the space of spinors V

$$\gamma = i^{s(s+1)/2}\gamma^1\gamma^2\dots\gamma^n. \tag{4.4}$$

Exercise 6. Show that γ defined by (4.4) is Hermitian and squares to the unit matrix I.

Note that since $\gamma^2 = I$, its eigenvalues are ± 1. In the cases when n is even, γ anticommutes with all the γ^a. It follows that each γ^a maps the $+1$ eigenspace of γ to the -1 eigenspace, and vice-versa. These eigenspaces are called the left-handed spinors and the right-handed spinors respectively, and they have equal dimension (except for the case $n = 0$).

In the cases where n is odd, γ commutes with all the γ^a, and so it has a constant eigenvalue in an irreducible representation. This is either $\gamma = 1$ or $\gamma = -1$, and this classifies the two inequivalent representations mentioned earlier. Thus in odd n, the spinors in an irreducible representation are either all left-handed or all right-handed.

4.3 Riemannian manifold

Let M be Riemannian, which means that is has a positive-definite metric g, and oriented, so there is a nowhere-zero n-form on M.

In local coordinates x on $U \subset M$, choose n vector fields e_1, e_2,..., e_n that form a positively-oriented orthonormal basis, which means that $g(e_a, e_b) = \delta_{ab}$. This is called a frame field. The dual basis of covectors is f^a, satisfying $f^a(e_b) = \delta^a_b$.

In these local coordinates, the spinor bundle is trivial, $S = U \times V$. The gamma matrices are type $(p, q) = (0, n)$, which means that $\eta^{ab} = \delta^{ab}$ in (4.2). The Clifford multiplication is $c(f^a, \psi) = \gamma^a\psi$. The tangent bundle has its usual Levi-Civita connection and it is assumed that this lifts to a connection on S, also denoted ∇, that intertwines the Clifford multiplication (i.e., the gamma matrices are covariantly constant).

If v is a vector field, then the notation ∇_v is used for the covariant derivative in the v direction, i.e., the 1-form from the action of ∇ contracted with v. One can write the identity operator on 1-forms as

$$I = f^a \otimes e_a \in T^*M \otimes T_*M \tag{4.5}$$

using the Einstein summation convention (which is the default notation where there is a repeated index). Thus

$$\nabla = f^a \otimes \nabla_{e_a}. \tag{4.6}$$

Using this notation, the Dirac operator (4.1) becomes

$$D\psi = \gamma^a \nabla_{e_a}\psi. \tag{4.7}$$

These local expressions can be pieced together over the whole manifold. If U and U' are domains of two coordinate charts, then

$$e'_a = \Lambda_a{}^b e_b \tag{4.8}$$

with the orthogonal matrices $\Lambda_a{}^b(x) \in \mathrm{SO}(n)$ the transition functions, for $x \in U \cap U'$. The transition functions for the spinor bundle are lifts of these functions to the group $\mathrm{Spin}_{\mathbb{C}}(n) = (\mathrm{Spin}(n) \times \mathrm{U}(1))/\mathbb{Z}_2$ provided by the action of $\mathrm{Spin}(n)$ in V which intertwines the action of $\mathrm{SO}(n)$ on the frame fields and the Clifford multiplication, and the action of $\mathrm{U}(1)$ on V, which commutes with all of the gamma matrices (because the gamma matrices are $\mathbb{C}$-linear). The data for lifting the transition functions to the $\mathrm{Spin}_{\mathbb{C}}$ group is called a $\mathrm{Spin}_{\mathbb{C}}$ structure for the manifold. This data isn't unique (and may not even exist, for $n > 4$).

Definition. The spectral triple for a compact Riemannian $\mathrm{Spin}_{\mathbb{C}}$ manifold is the triple $(\mathcal{A}, \mathcal{H}, D)$ where

- $\mathcal{A} = C^\infty(M)$ is the algebra of smooth functions on M
- $\mathcal{H} = L^2(M, S)$ is the Hilbert space of sections of the spinor bundle, with inner product $\langle \psi, \phi \rangle = \int_M \langle \psi(x), \phi(x) \rangle \, \mathrm{d}V_x$
- $D \colon \mathcal{H} \to \mathcal{H}$ is the Dirac operator.

From this data it is possible to reconstruct the manifold with its metric and Dirac operator. Connes has a set of axioms for spectral triples $(\mathcal{A}, \mathcal{H}, D)$ so that they are the data for a compact Riemannian $\mathrm{Spin}_{\mathbb{C}}$ manifold [9, Thm 11.5]. This theorem can be viewed as an analogue of Gelfand's representation theorem for Riemannian manifolds.

Chapter 5

Real structures

5.1 Real algebras

A real structure on a Hilbert space is an antilinear operator $J\colon \mathcal{H} \to \mathcal{H}$ which squares to ϵI, with $\epsilon = \pm 1$. A bit confusingly, the case $J^2 = I$ is called a real type (of real structure) and $J^2 = -I$ a quaternionic type. These real structures can arise when there is a real algebra or real Lie group acting on a complex vector space, usually because the real structure commutes with the action.

A real algebra is a vector space over $\mathbb{R}$ with a multiplication law, but it can still act on a complex vector space. For example, $x \in \mathbb{R}$ acts on $\mathbb{C}$ by $c \mapsto xc$. The real structure $J = *$ (complex conjugation) commutes with the action of x, i.e.,

$$J(xc) = x(Jc). \tag{5.1}$$

The second example of a real algebra is the quaternions $\mathbb{H}$. This is the real algebra of 2×2 matrices that are real linear combinations of

$$q_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad q_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad q_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$q_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{5.2}$$

A real structure for this algebra is an antilinear map on $\mathbb{C}^2$,

$$J \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \beta^* \\ -\alpha^* \end{pmatrix} \tag{5.3}$$

This is of quaternionic type, since $J^2 = -I$. In both cases, J characterises the algebra: $\mathbb{R}$, respectively $\mathbb{H}$, is the algebra of all the matrices that commute with J.

5.2 Reality for gamma matrices

Gamma matrices also have a real structure. For a given type (p, q) of matrices, there is a real structure that satisfies

$$J\gamma^a = \epsilon' \gamma^a J \tag{5.4}$$

s	0	1	2	3	4	5	6	7
ϵ	1	1	-1	-1	-1	-1	1	1
ϵ'	1	-1	1	1	1	-1	1	1
ϵ''	1	1	-1	1	1	1	-1	1
	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}$	$\mathbb{H}$	$\mathbb{C}$	$\mathbb{R}$	$\mathbb{R}$

Figure 5.1: Table of signs for gamma matrices and real spectral triples.

for every a, where $\epsilon' = \pm 1$, independently of a. The value of ϵ and ϵ' depend only on the value of $s = q - p \mod 8$, as shown in Figure 5.1.

As an example, consider $(p, q) = (0, 2)$. One choice for the gamma matrices is the quaternions $\gamma^1 = q_1$, $\gamma^2 = q_2$. Then the real structure is the one for the quaternions in (5.3).

One way of understanding the real structure for the type (p, q) gamma matrices is to work out the algebra generated by sums and products of the gamma matrices. Since it was assumed that the representation is irreducible, this algebra must be isomorphic to one of the three real algebras $M_m(\mathbb{R})$, $M_m(\mathbb{C})$ or $M_m(\mathbb{H})$ for a suitable value of m. (For $\mathbb{H}$, substituting 2×2 matrices for the quaternions gives elements of $M_{2m}(\mathbb{C})$.) Some of these are shown in the table in Figure 5.2.

	$q=0$	1	2	3	4	$\ldots$
$p=0$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}$	$M_2(\mathbb{H})$	$\ldots$
1	$\mathbb{R}$	$M_2(\mathbb{R})$	$M_2(\mathbb{C})$	$M_2(\mathbb{H})$	$\ldots$	
2	$M_2(\mathbb{R})$	$M_2(\mathbb{R})$	$M_4(\mathbb{R})$	$\ldots$		
3	$M_2(\mathbb{C})$	$M_4(\mathbb{R})$	$\ldots$			
4	$M_2(\mathbb{H})$	$\ldots$				
$\vdots$						

Figure 5.2: The algebras generated by irreducible gamma matrices.

One can see that the use of $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ depends only on the parameter s (as indicated on the bottom of Figure 5.1). In the real and quaternionic cases, J is just the real structure for these algebras. However the complex algebras $M_m(\mathbb{C})$ have no real structure. In these cases, the real structure is the one for the algebra generated by $i\gamma^a$, which is in fact real or quaternionic, belonging to the case where p and q are swapped. In these complex cases, $\epsilon' = -1$.

The third sign in the table is $\epsilon'' = \pm 1$ in the equation

$$J\gamma = \epsilon''\gamma J. \tag{5.5}$$

This sign can be calculated from the definition of the chirality in (4.4).

Exercise 7. Check the values of ϵ'' in Figure 5.1, starting from the definition (4.4).

5.3 Spin structure

In Ch 4.3 it was explained that a Riemannian $\mathrm{Spin}_{\mathbb{C}}$ manifold can be expressed as a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ consisting of a commutative algebra, a Hilbert space and a Dirac operator. However, $\mathrm{Spin}_{\mathbb{C}}$ manifolds are somewhat too general for known physics since the 'gauge group' $\mathrm{Spin}_{\mathbb{C}}(n) = (\mathrm{Spin}(n) \times \mathrm{U}(1))/\mathbb{Z}_2$ contains an unwanted U(1) factor.

The solution to this problem is to add a real structure $J\colon \mathcal{H} \to \mathcal{H}$ to the spectral triple data. In local coordinates, the real structure acts pointwise, i.e. $(J\psi)(x) = J(\psi(x))$, with the J on the right being the real structure for spinors discussed above.

The local expressions for J have to glue together correctly to give J for the whole manifold. The operator J commutes with transition functions in the group $\mathrm{Spin}(n)$ because its Lie algebra is generated by

$$\gamma^a\gamma^b - \gamma^b\gamma^a \tag{5.6}$$

and J commutes with these in every case. However J does not commute with U(1), $J\exp(i\theta) = \exp(-i\theta)J$, so the transition functions have to be in the group $\mathrm{Spin}(n)$, not $\mathrm{Spin}_{\mathbb{C}}$. Likewise, the connection has to be a $\mathrm{Spin}(n)$ connection. Such a manifold is called a Riemannian Spin manifold. It is the curved-space generalisation of the usual theory of spinors in flat space.

Chapter 6

Real spectral triples

The real structure was an important development in spectral triples, first expounded in [6]. It enabled the correct development of the non-commutative case. The axioms for spectral triples were first listed in [7], distinguishing the commutative and non-commutative cases.

These axioms have developed a little in time: one axiom (Poincare duality) was dropped, as it was discovered to be unnecessary and in conflict with desirable examples, while two other axioms had slight technical modifications to allow the proof of the reconstruction theorem in [9].

The axioms for a real spectral triple $(s, \mathcal{A}, \mathcal{H}, D, J)$ are given here. First, there are the axioms that apply generally, then the axioms specific to the non-commutative and commutative cases are added separately.

6.1 General axioms

There is a parameter s which is an integer modulo 8. This parameter is often called the KO-dimension of the spectral triple (due to the similar role it plays in K-homology).

There is a Hilbert space $\mathcal{H}$, with its Hermitian inner product denoted $\langle \cdot\,, \cdot \rangle$. The Hilbert space can be finite or infinite-dimensional. In the former case, the spectral triple is called a finite spectral triple.

There is a chirality operator $\gamma\colon \mathcal{H} \to \mathcal{H}$ obeying $\gamma = \gamma^*$, $\gamma^2 = I$. The Hilbert space is called the space of fermions, and the $+1$ and -1 eigenspaces of γ are called the left- and right-handed fermions. This chirality operator is sometimes regarded as part of the structure of the Hilbert space (summarised by saying that $\mathcal{H}$ is a $\mathbb{Z}_2$-graded Hilbert space).

There is a $*$-algebra $\mathcal{A}$ (an algebra over either $\mathbb{R}$ or $\mathbb{C}$) and a faithful left action of $\mathcal{A}$ in $\mathcal{H}$ by bounded operators that commute with γ. The action of $a \in \mathcal{A}$ is the operator denoted $l(a)$. This action obeys the condition that $l(a^*) = l(a)^*$, the adjoint of the operator $l(a)$.

Note that many references define $\mathcal{A}$ as an algebra of operators in $\mathcal{H}$ directly, in which case a and $l(a)$ are actually the same thing. However, this can be confusing in places, so the distinction between a and $l(a)$ has been kept. Here, the notation $a\psi$ is reserved for the multiplication of matrices in the cases where that makes sense.

There is an operator $D\colon \mathcal{H} \to \mathcal{H}$, called the Dirac operator, that is self-adjoint, $D = D^*$. It obeys $D\gamma = -\gamma D$ in the s even case, which means that it maps left-handed fermions to right-handed ones, and vice-versa.

In the s odd case, it obeys $D\gamma = \gamma D$, so that chirality is preserved. In fact, in this odd case, one can decompose $\mathcal{H}$ into its two eigenspaces of γ and both of them form spectral triples. So it is often said that the chirality is trivial in the odd case, and it can be assumed this means that $\gamma = I$.

There is an antilinear operator $J\colon \mathcal{H} \to \mathcal{H}$ that is unitary in the sense that $\langle J\psi, J\phi\rangle = \langle \psi, \phi\rangle^*$. This obeys $J^2 = \epsilon$, $JD = \epsilon' DJ$, $J\gamma = \epsilon''\gamma J$ according to the signs in Figure 5.1.

From the axioms so far, there is a second action of $\mathcal{A}$ in $\mathcal{H}$, given by

$$r(a) = Jl(a)^* J^{-1} \tag{6.1}$$

It is a right action, because $r(ab) = r(b)r(a)$. The treatment of this differs between the commutative and the noncommutative cases.

6.2 Commutative axioms

Here, $\mathcal{A}$ is a commutative algebra of functions on some space M. It is simplest to assume that the two actions of $\mathcal{A}$ in $\mathcal{H}$ are in fact the same, i.e.,

$$l(a) = Jl(a)^* J^{-1}, \tag{6.2}$$

so there is only one action, as is the case for the differential geometry of a manifold. The equation (6.2) can also be read as saying that J acts pointwise on M (assuming $*$ is pointwise complex conjugation).

The Dirac operator on a manifold is a first-order differential operator. This is coded in the axioms by requiring

$$[[D, l(a)], l(b)] = 0 \tag{6.3}$$

for all $a, b \in \mathcal{A}$. This is called the *first-order condition.* On a manifold,

$$[D, l(f)] = [\gamma^a \nabla_{e_a}, f] = \gamma^a e_a(f) = c(\mathrm{d}f), \tag{6.4}$$

which is a zeroth-order differential operator (it does not differentiate the fermion fields). This commutes with multiplication by another function.

The remainder of the axioms in [9] are aimed at characterising a smooth manifold with a specified dimension

n. These axioms are summarised here very briefly, as they are not used in the following.

Firstly the operator D^{-1} (after removing zero-eigenvectors of D) is required to be a compact operator. This means that D has a discrete spectrum and (in the infinite case) it is unbounded. This condition is vacuous in the finite case. The m-th eigenvalue is required to grow no faster than $|\lambda_m| \sim m^{1/n}$, which is characteristic of the Dirac operator on an n-manifold.

There is an axiom that ensures that the algebra $\mathcal{A}$ is an algebra of smooth functions (with respect to differentiation by D). There is another axiom that $\mathcal{H}$ is a bundle over the space with finitely many coordinates from $\mathcal{A}$.

Finally, there is an axiom that says that the chirality operator can be obtained from a finite sum of n 'differentiations' of functions in $\mathcal{A}$. The formula is

$$\gamma = \sum_j a_j^0 [D, a_j^1][D, a_j^2] \ldots [D, a_j^n]. \tag{6.5}$$

On a manifold, each commutator should give the gradient of the function contracted with gamma matrices, as in (6.4), and so this axiom is understood as ensuring that the chirality operator is the product of n gamma matrices. The sum is supposed to be antisymmetric in the a^1 to a^n, and so this formula can also be viewed as the Clifford multiplication of the volume form.

6.3 Non-commutative axioms

In the non-commutative case, the Hilbert space has a left action l and a right action r. The real structure intertwines them, $l(a)J = Jr(a)^*$ and likewise $r(a)J = Jl(a)^*$.

It does not make sense to set the two actions equal, because this would imply that the algebra is commutative. The next best thing is to require that the actions commute,

$$[l(a), r(b)] = 0 \tag{6.6}$$

for all a, b. This equation is called the *zeroth-order condition* because in the commutative case it would guarantee that $l(a)$ (or $r(b)$) is a zeroth-order differential operator. A Hilbert space with commuting left and right actions is called a *bimodule* over the algebra.

The other non-commutative axiom to give here is a replacement for (6.3). The condition written there does not make sense for a non-commutative algebra. The Jacobi identity reads

$$[[D, l(a)], l(b)] + [l(a), [D, l(b)]] = [D, [l(a), l(b)]] \tag{6.7}$$

so D would have to commute with all commutators, which is way too restrictive.

The fruitful generalisation is the first-order condition in the non-commutative case

$$[[D, l(a)], r(b)] = 0 \tag{6.8}$$

for all a, b. The Jacobi identity now reads

$$[[D, l(a)], r(b)] + [l(a), [D, r(b)]] = [D, [l(a), r(b)]] = 0 \tag{6.9}$$

so it is not restrictive. It also shows that interchanging left and right in (6.8) gives the same thing.

Finally, what of the remaining axioms from the commutative case? It isn't clear what the analogues of these should be, since there is no general definition of a non-commutative n-manifold. A useful definition will only become apparent through the examination of sufficiently many interesting examples.

Chapter 7

Standard model charges

The standard model of particle physics was one of the motivating examples for the development of spectral triples. Non-commutative geometry applies to understanding the geometry of the classical fields in the model. The success of the theory is that it provides a tighter mathematical structure for the known fields and explains (at least in a mathematical sense) some of its features. It does not (yet!) provide any new sort of quantum field theory. However it is interesting because it provides some new departure points for attempts to develop new quantum theory.

The first recognisably modern version of the spectral triple by Connes came in 1995 with the development of the real structure [6]. The idea behind the construction is to think of the geometry as the cartesian product of the usual (commutative) space-time four-manifold with a non-commutative 'internal space'. There is a spectral triple for the space-time manifold and a second spectral triple for the internal space. The complete geometry is obtained by a suitable non-commutative version of the Cartesian product, which is a tensor product of spectral triples.

The theory explained by the Connes theory is for Riemannian (i.e., positive signature) metrics. This is not immediately physical, because known physics is in a Lorentzian signature space-time. However, that is technically awkward because the Hermitian form on spinors (and hence on spinor fields) is not positive definite and so one does not have a Hilbert space of the classical fermion fields. It is known in quantum field theory that there is a close relation between physics on Minkowski space and 'physics' on Euclidean space, by way of Wick rotation of the quantum field theories. Thus it is not too far-fetched to consider physics on an arbitrary (compact) Riemannian space-time, and these notes will concentrate on that case for its technical simplicity.

The general idea is similar to the Kaluza-Klein attempts at unifying space-time geometry with matter fields by adding extra dimensions. In the original version of that theory, there is a five-dimensional space-time which is a product of a four manifold and a circle, the simplest version of the internal space. If the vector field around the circle is assumed to be a Killing vector, the metric tensor can be written as a function of the four space-time coordinates $x = (x^1, x^2, x^3, x^4)$, and decomposes into $g_{\mu\nu}(x)$, $g_{\mu 5}(x)$ and $g_{55}(x)$ components, which are reinterpreted (after some changes of variables) as a four-dimensional metric, an electromagnetic field and a scalar field.

More modern versions of Kaluza-Klein allow an internal space of higher dimension (six dimensions is popular) and obtain both non-abelian gauge fields and Higgs scalar fields. However, the Kaluza-Klein models suffer from various non-physical features and the necessity of ad hoc assumptions that are unnatural for the geometry. There is also, in principal, an infinite 'tower' of harmonics on the internal space each of which could be regarded as a field in x but have to be explained away as currently unobservable.

The non-commutative version of the internal space is much better than the commutative version because of the possibility of finite spectral triples, that is, ones where the Hilbert space is finite-dimensional. The only way to do this in Riemannian geometry would be to have an internal space M with a finite set of points, so that $L^2(M)$ is finite-dimensional, but then the Riemannian metric of this space becomes trivial.

7.1 Internal space

The internal space of the standard model is constructed as follows. The algebra of the spectral triple is the real algebra

$$\mathcal{A} = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}). \tag{7.1}$$

The summands $\mathbb{C}$ and $M_3(\mathbb{C})$ are regarded as vector spaces over $\mathbb{R}$, of dimension 2 and 18. The $*$ is the usual complex conjugation and Hermitian conjugation of matrices. A typical element will be written $a = (c, q, m)$.

The algebra contains unitary elements $a^* = a^{-1}$ forming the Lie group $U = \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{U}(3)$. This is almost the gauge group of the standard model, so in a sense, some of the structure of the standard model has been put in 'by hand' in the choice of algebra.

The algebra has two representations in $M_4(\mathbb{C})$ as the block matrices

$$\pi_1(a) = \begin{pmatrix} q & \cdot & \cdot \\ \cdot & c & \cdot \\ \cdot & \cdot & c^* \end{pmatrix}, \qquad \pi_2(a) = \begin{pmatrix} c & \cdot \\ \cdot & m \end{pmatrix} \tag{7.2}$$

These act on the Hilbert space $\mathfrak{h} = M_4(\mathbb{C})$, with left action $\psi \mapsto \pi_1(a)\psi$ and right action $\psi \mapsto \psi\pi_2(a)$, making $\mathfrak{h}$ a bimodule over $\mathcal{A}$.

To understand this physically, matrix elements of $\mathfrak{h}$ are given names as physical fields.

$$\psi = \begin{pmatrix} \nu_L & u_L^1 & u_L^2 & u_L^3 \\ e_L & d_L^1 & d_L^2 & d_L^3 \\ \nu_R & u_R^1 & u_R^2 & u_R^3 \\ e_R & d_R^1 & d_R^2 & d_R^3 \end{pmatrix} \tag{7.3}$$

One should think of $\mathfrak{h}$ as the space of all fermion fields of one generation of the standard model but ignoring the spinor indices and the dependence on space-time. It is convenient to collect them all up as one mathematical object rather than thinking of them as several separate fields.

The notation is ν for the neutrino, e for the electron, u for the up quark and d for the down quark, each of which has three components for the three different colours of quarks. Each field has a left-handed and a right-handed basis vector because in the standard model, the left- and right-handed fields have quite different charges. There is a right-handed neutrino, so this is, strictly speaking, the standard model augmented with a right-handed neutrino. From this matrix perspective, it would be strange to leave it out. Note that the three generations of the standard model can be obtained as the space $\mathfrak{h} \otimes \mathbb{C}^3$.

The chirality operator $\gamma_{\mathfrak{h}} \colon \mathfrak{h} \to \mathfrak{h}$ is

$$\psi \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \psi = \chi\psi \tag{7.4}$$

which commutes with both the left and right actions of $\mathcal{A}$.

The gauge group emerges from considering the adjoint action of the group U. The action of $u \in U$ is

$$\rho(u)\psi = \pi_1(u)\,\psi\,\pi_2(u^{-1}) \tag{7.5}$$

From this, one can read off the irreducible representations of the group U.

However, this is not exactly the right group for the standard model. The correct gauge group $G \subset U$ emerges from the 'unimodularity condition' $\det \pi_2(u) = 1$. Note that $\det \pi_1(u) = 1$ automatically.

Writing $m \in \mathrm{U}(3)$ as $m = ts$ with $s \in \mathrm{SU}(3)$ and $t \in \mathrm{U}(1)$, the condition is

$$\det \pi_2(u) = t^3 c = 1. \tag{7.6}$$

One can set $c = t^{-3}$, though it is standard in the physics literature to write $t = c^{-1/3}$ and eliminate t instead, at the expense of having charges that are a multiple of 1/3

	U(1)	SU(2)	SU(3)
$\begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$	$t^3 = c^{-1}$	q	I
$\begin{pmatrix} u_L^1 & u_L^2 & u_L^3 \\ d_L^1 & d_L^2 & d_L^3 \end{pmatrix}$	$t^{-1} = c^{1/3}$	q	$(s^{-1})^T$
ν_R	1	I	I
e_R	$t^6 = c^{-2}$	I	I
$\begin{pmatrix} u_R^1 & u_R^2 & u_R^3 \end{pmatrix}$	$t^{-4} = c^{4/3}$	I	$(s^{-1})^T$
$\begin{pmatrix} d_R^1 & d_R^2 & d_R^3 \end{pmatrix}$	$t^2 = c^{-2/3}$	I	$(s^{-1})^T$

Figure 7.1: Table of irreducible representations of the gauge group G. The power of c is the weak hypercharge.

(which just indicate that one should really consider t as parameterising U(1)). Thus, the group $G = \{(t, q, s)\} = \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$ acts on the fermions, via the adjoint action of $(c, q, m) = (t^{-3}, q, ts)$.

The irreducible representations of G are shown in Figure 7.1. Matrix transpose has been used to convert a right action on a row vector to a left action on a column vector and the unit matrix I is the trivial representation.

These are the correct assignment of charges to the standard model fields, taking the power of c to be the weak hypercharge. The representation of SU(3) comes out to be $(s^{-1})^T = \overline{s}$, which is the complex conjugate of the matrix s. This complex conjugation is not significant; one could instead start with the definition $m = t\overline{s}$. Then the action on the quarks is with the matrix $(\overline{s}^{-1})^T = s$, so that the quarks are in the fundamental of SU(3) rather than its complex conjugate.

Arriving at the correct standard model representations is a very strong result. Algebras have very few representations. The algebra $\mathbb{H}$ has only one representation (up to equivalence) and $\mathbb{C}$ or $M_3(\mathbb{C})$ have only two: the fundamental and the complex conjugate of it. For a direct sum of simple algebras, a representation is just one of these representations for one of the simple summands. There isn't even a trivial representation.

This is in complete contrast to the situation with Lie groups, which have an infinite set of representations, which can be tensored together arbitrarily for a product group. So it is a startling result that all of the fermion charges can be obtained from a bimodule over an algebra.

The status of the unimodular condition is that it (or something like it) appears to be necessary in quantum field theory to avoid gauge anomalies [5]. Whether it has a good explanation in non-commutative geometry itself is a good question.

Exercise 8. Check that the representations in Table 7.1 are correct.

Chapter 8

Standard Model masses

8.1 Real structure

The Hilbert space $\mathfrak{h}$ constructed in Ch 7 does not have a real structure for the bimodule action of $\mathcal{A}$. A related problem with the physics is that it is not possible to express Majorana mass terms in terms of complex fields only.

To solve these problems, it is necessary to have the charge conjugates of the fields in the formalism as independent variables. Then one can construct the internal space as a finite real spectral triple with KO-dimension

$s = 6$. This was done from the point of view of Lorentzian fields [2] and also from the point of view of Euclidean fields [8], giving the same internal space.

This construction for one generation of fermions has Hilbert space $\mathcal{H}_G = M_4(\mathbb{C}) \oplus M_4(\mathbb{C}) \subset M_8(\mathbb{C})$ arranged as an off-diagonal block matrix

$$\Psi = \begin{pmatrix} \cdot & \psi \\ \overline{\psi} & \cdot \end{pmatrix} \tag{8.1}$$

with ψ and $\overline{\psi}$ independent 4×4 matrices. This vector space has dimension 32.

The algebra action can be packaged into block diagonal 8×8 matrices, as

$$\pi(a) = \begin{pmatrix} \pi_1(a) & \cdot \\ \cdot & \pi_2(a) \end{pmatrix} \tag{8.2}$$

The left action of $\mathcal{A}$ on $\mathcal{H}_G$ is left multiplication by $\pi(a)$

$$l(a)\Psi = \pi(a)\Psi = \begin{pmatrix} \cdot & \pi_1(a)\psi \\ \pi_2(a)\overline{\psi} & \cdot \end{pmatrix} \tag{8.3}$$

and the right action is right multiplication,

$$r(a)\Psi = \Psi\pi(a) = \begin{pmatrix} \cdot & \psi\pi_2(a) \\ \overline{\psi}\pi_1(a) & \cdot \end{pmatrix}. \tag{8.4}$$

For the top right-hand block, this is the same as the bimodule $\mathfrak{h}$.

The real structure $J\colon \mathcal{H}_G \to \mathcal{H}_G$ is just the Hermitian conjugate of matrices, $J\Psi = \Psi^*$. It satisfies

$$J^2 = 1. \tag{8.5}$$

Exercise 9. Check that J intertwines the left and right actions of $\mathcal{A}$ on $\mathcal{H}_G$ according to (6.1).

One can use the real structure to name the matrix entries in the lower left block.

$$\overline{\psi} = \begin{pmatrix} \overline{\nu}_L & \overline{e}_L & \overline{\nu}_R & \overline{e}_R \\ \overline{u}^1_L & \overline{d}^1_L & \overline{u}^1_R & \overline{d}^1_R \\ \overline{u}^2_L & \overline{d}^2_L & \overline{u}^2_R & \overline{d}^2_R \\ \overline{u}^3_L & \overline{d}^3_L & \overline{u}^3_R & \overline{d}^3_R \end{pmatrix} \tag{8.6}$$

Note that these matrix components are *independent* of the components of ψ. Obviously one could consider elements of $\mathcal{H}_G$ that satisfy $J\Psi = \Psi$, and then it would be true that $\overline{\nu}_L = (\nu_L)^*$, and similarly for the other matrix elements. But in general, this does not hold.

The chirality operator $\gamma\colon \mathcal{H}_G \to \mathcal{H}_G$ is

$$\gamma\Psi = \begin{pmatrix} \chi & \cdot \\ \cdot & I \end{pmatrix} \Psi \begin{pmatrix} -\chi & \cdot \\ \cdot & I \end{pmatrix} = \begin{pmatrix} \cdot & \chi\psi \\ -\overline{\psi}\chi & \cdot \end{pmatrix} \tag{8.7}$$

with χ the 4×4 matrix in (7.4). This means that the components labelled L in ψ are left-handed, and R right-handed. But for the components of $\overline{\psi}$ it is the other way around: $\overline{\nu}_L$ is right-handed, for example. This is summed up by the relation

$$J\gamma = -\gamma J \tag{8.8}$$

which, together with (8.5), characterises KO-dimension 6.

The standard model has three generations of fermions. These are three copies of the Hilbert space constructed so far, with exactly the same structures. Thus the full Hilbert space is

$$\mathcal{H} = \mathcal{H}_G \otimes \mathbb{C}^3 = \mathcal{H}_G \oplus \mathcal{H}_G \oplus \mathcal{H}_G \tag{8.9}$$

and it has dimension $3 \times 32 = 96$.

The first attempt for a real spectral triple for the internal space in [6] was constructed in the same way but with the opposite chirality for $\overline{\psi}$, so that $J\gamma = \gamma J$. This meant that the KO-dimension was 0. This affected the allowed Dirac operators (as explained in the next section) and the results did not correspond well to the physics. The chirality operator was corrected in [2] and [8].

8.2 Dirac operators

For finite spectral triples in general, every Dirac operator splits

$$D = D_R + D_L, \tag{8.10}$$

where $[D_R, r(a)] = 0$ and $[D_L, l(a)] = 0$ for all $a \in \mathcal{A}$. Then the first order condition (6.8) is satisfied automatically. This principle was discovered by Krajewski [11] for some special cases and proved in a general form in [3, Lemma 3].

The splitting can be chosen such that

- $D_R = D_R^*$
- $D_R\gamma = \begin{cases} -\gamma D_R & (s \text{ even}) \\ \gamma D_R & (s \text{ odd}) \end{cases}$
- $D_L = \epsilon' J D_R J^{-1}$.

Exercise 10. Show that if D_L, D_R have these properties, then D is a Dirac operator.

Thus to understand the possible Dirac operators, one just has to construct D_R. This can be written as a finite sum of terms, each of the form

$$D_R\Psi = M\Psi P \tag{8.11}$$

where P is a matrix that commutes with the representation of the algebra by π_2. The other two conditions on D_R have to be checked.

This general theory can be applied to the internal space of the standard model. For example, if P_q is the projector onto the representations $(c, q, m) \mapsto m$, i.e.,

$$P_q = \begin{pmatrix} 0_5 & \cdot \\ \cdot & I_3 \end{pmatrix}, \tag{8.12}$$

then

$$\Psi P_q = \left(\begin{array}{ccccc|cccc} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & u_L^1 & u_L^2 & u_L^3 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & d_L^1 & d_L^2 & d_L^3 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & u_R^1 & u_R^2 & u_R^3 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & d_R^1 & d_R^2 & d_R^3 \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right) \tag{8.13}$$

Since P_q commutes with chirality, it must be that M_q anti-commutes with it. Therefore the matrix M_q is determined by a 2×2 matrix N_q, written in 2×2 blocks

$$M_q = \left(\begin{array}{cc|cc} \cdot & N_q & \cdot & \cdot \\ N_q^* & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array}\right) \tag{8.14}$$

This term in the Dirac operator gives

$$D_R\Psi = M_q(\Psi P_q) = \left(\begin{array}{c|cc}
\begin{matrix}\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\end{matrix} & \begin{matrix}\cdot\\ \cdot\end{matrix} & N_q\begin{pmatrix}u_R^1 & u_R^2 & u_R^3\\ d_R^1 & d_R^2 & d_R^3\end{pmatrix}\\
\begin{matrix}\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\end{matrix} & \begin{matrix}\cdot\\ \cdot\end{matrix} & N_q^*\begin{pmatrix}u_L^1 & u_L^2 & u_L^3\\ d_L^1 & d_L^2 & d_L^3\end{pmatrix}\\ \hline
\begin{matrix}\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\end{matrix} & \begin{matrix}\cdot\\ \cdot\\ \cdot\\ \cdot\end{matrix} & \begin{matrix}\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot\end{matrix}
\end{array}\right) \tag{8.15}$$

which can be seen as mass terms for quarks. This interpretation is explained further in Chapter 11.

Exercise 11. Show that if P is the projector onto the representation q or c^* then there is no contribution to D_R.

The case of the representation c is a bit more complicated because it appears in (8.2) with multiplicity two. This means the possible P are parameterised by a 2×2 matrix.

$$P = \begin{pmatrix} 0_2 & \cdot & \cdot & \cdot & \cdot\\ \cdot & a_{11} & \cdot & a_{12} & \cdot\\ \cdot & \cdot & 0 & \cdot & \cdot\\ \cdot & a_{21} & \cdot & a_{22} & \cdot\\ \cdot & \cdot & \cdot & \cdot & 0_3 \end{pmatrix}. \tag{8.16}$$

Exercise 12. Show that one choice of P in (8.16) and a suitable M gives lepton mass terms directly analogous to the quark mass terms above. Show that different choices of P and M couple ν_R or e_R with $\overline{\nu}_R$ or the three $\overline{u}^i_R$.

Note that the coupling of ν_R and $\overline{\nu}_R$ is a Majorana mass term for the right-handed neutrino. This is a highly valuable feature of the theory since it allows for realistic neutrino masses via the 'seesaw mechanism' of high-energy physics.

The possibility of lepton couplings to the quark fields is not desirable, and compatibility with physics requires these matrix entries in the Dirac operator to be zero. This is perfectly consistent with the non-commutative geometry, but is not explained by it.

Chapter 9

Fuzzy sphere spectral triple

The purpose of this chapter is to discuss the fuzzy sphere as a spectral triple. The fuzzy sphere was introduced in Ch 2 using the Laplace operator. The Dirac operator takes more work to describe but is ultimately a more flexible concept.

The chapter starts with a description of the commutative Dirac operator and a computation of its spectrum. The fuzzy case is very similar, and just appears to be a cut-off version of the commutative case, as was true for the Laplacian.

9.1 Commutative Dirac operator

The sphere has three standard vector fields on it, V_1, V_2, V_3, as in (3.3), satisfying $[V_1, V_2] = V_3$ and cyclic permutations of this equation. Since the sphere is two-dimensional, the formula (4.7) for the Dirac equation would require two orthonormal vector fields, but that is awkward. Instead it is better to generalise the formula, at the expense of introducing more gamma matrices.

The following construction is quite general. Suppose M is an n-manifold. In local coordinates with n orthonormal vector fields e_a the inverse metric g is

$$g = \eta^{ab} e_a \otimes e_b, \tag{9.1}$$

using the constant coefficients η^{ab} from (4.2) (which are equal to δ^{ab} for a Riemannian manifold).

One can add m further gamma matrices $\gamma^{n+1}, \dots, \gamma^{n+m}$, enlarging the spinors if necessary (they double in size for every two extra gamma matrices). Setting $e_{n+1} = e_{n+2} = \dots = 0$, the same formulas can be used for the Dirac operator (4.7) and inverse metric (9.1), but summing over $a, b = 1, \dots, n+m$, with the bigger matrix η'^{ab}. In this formula, the new gamma matrices are redundant.

Now define new vector fields $e'_a = \Lambda_a{}^b e_b$, using an x-dependent orthogonal transformation $\Lambda(x) \in \mathrm{SO}(n+m)$.

Then the $n+m$ vector fields e'_a are not orthonormal, but they do still satisfy

$$\eta'^{ab} e'_a \otimes e'_b = \eta'^{ab} \Lambda_a{}^c e_c \otimes \Lambda_b{}^d e_d = \eta'^{ab} e_a \otimes e_b = g. \quad (9.2)$$

The Dirac operator becomes

$$D = \gamma^a (\Lambda^{-1})_a{}^b \nabla_{e'_b}. \quad (9.3)$$

The matrix Λ^{-1} can be removed using the intertwining with $h^{-1} \in \mathrm{Spin}(n+m)$,

$$\gamma^a (\Lambda^{-1})_a{}^b = h^{-1} \gamma^b h \quad (9.4)$$

so that

$$D\psi = h^{-1}\gamma^b h \nabla_{e'_b} \psi = h^{-1}\gamma^b \nabla'_{e'_b} h\psi \quad (9.5)$$

This shows that $\psi' = h\psi$ is a gauge transformation which, after removing the primes, returns the Dirac equation to the same formula (4.7) but with $n+m$ vector fields and gamma matrices.

In the context of the sphere, this means one can write the Dirac operator as

$$D = \gamma^1 \nabla_{V_1} + \gamma^2 \nabla_{V_2} + \gamma^3 \nabla_{V_3} \quad (9.6)$$

with three 2×2 gamma matrices of type $(0,3)$. This formula is in fact global, as the Dirac spinor bundle for a sphere is the trivial vector bundle. The KO-dimension for these gamma matrices is $s = 3$, and it is convenient to use the spinors for which $\gamma = \gamma^1\gamma^2\gamma^3 = I$, so that $\gamma^1 = -\gamma^2\gamma^3$, and cyclic permutations.

It is somewhat shocking that a 2-dimensional sphere can end up with spinors with $s = 3$. However, there are a number of examples where s does not match the dimension of the space on geometric grounds. For example, the spectral triple for the standard model in Ch 8 has $s = 6$, whereas one might expect that a finite spectral triple is a non-commutative 0-manifold (the only sorts of finite spectral triple in commutative Riemannian geometry) and so should have $s = 0$.

The difference between type $(0, 3)$ spinors and type $(0, 2)$ spinors is that the $(0, 3)$ spinors do not have a non-trivial chirality operator. However, at each point $r \in S^2$, one can define a chirality operator for the tangent space T_rS^2 as $\gamma_r = r_1\gamma^1+r_2\gamma^2+r_3\gamma^3$, giving a type $(0, 2)$ Clifford algebra for each tangent space (generated by the vectors orthogonal to r). This construction varies with r, and in fact the line bundles of left-handed or right-handed spinors are non-trivial vector bundles over S^2. A chirality operator that varies from point to point does not make sense directly in non-commutative geometry, so the fact that one can have $s = 2$ by a local construction does not lead directly to a non-commutative version.

The covariant derivative at the point $r = (0, 0, 1)$ is determined by

$$\begin{aligned} \nabla_{V_1} V_a &= [V_1, V_a] \\ \nabla_{V_2} V_a &= [V_2, V_a] \\ \nabla_{V_3} V_a &= 0, \end{aligned} \tag{9.7}$$

the last one because $V_3 = 0$ at $(0, 0, 1)$. The other two vector fields at that point are $V_1 = \partial_2$ and $V_2 = -\partial_1$, which are an orthonormal basis.

Exercise 13. Show that the torsion $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$ is equal to 0 for all tangent vectors at $(0, 0, 1)$. Also compute the covariant derivative of the inverse metric (9.2) at $(0, 0, 1)$.

The formula can be generalised to any point r by taking appropriate linear combinations of the vector fields.

$$\nabla_{V_b} V_a = [V_b, V_a] - r_b r^c [V_c, V_a], \tag{9.8}$$

writing $r^c = \delta^{cd} r_d$. All index sums are over three values, $1, 2, 3$.

An arbitrary vector field can be expressed as $X = f^a V_a$, where f^a are functions on S^2. Using the Liebnitz rule, the covariant derivative of X is

$$\nabla_{V_b} X = (V_b f^a) V_a + f^a \nabla_{V_b} V_a \tag{9.9}$$

The first term is a derivative operator on the f^a, while the second term is the multiplication by connection coefficients.

The covariant derivative on spinors is obtained by replacing the rotation generators $[V_a, \cdot\,]$ by the corresponding spin generators. The spin generators are $S_1 = \frac{1}{2}\gamma^2\gamma^3 = -\frac{1}{2}\gamma^1$ and cyclic permutations, so that $S_a = -\frac{1}{2}\gamma^a$. These satisfy

$$[S_1, S_2] = S_3 \tag{9.10}$$

etc., the same as the V_a, and also $[V_a, \cdot\,]$. In other words, these generate the Lie algebra of SU(2) in the spin representation.

For example, at $(0, 0, 1)$ one gets

$$\begin{aligned}\nabla_{V_1}\psi &= V_1\psi + S_1\psi\\ \nabla_{V_2}\psi &= V_2\psi + S_2\psi\\ \nabla_{V_3}\psi &= 0.\end{aligned} \tag{9.11}$$

acting on $\mathcal{H} = \mathbb{C}^2 \otimes L^2(S^2)$, with the first terms V_a just differentiating the spinor components as functions on S^2. The Dirac operator (9.6) at $(0, 0, 1)$ is

$$\begin{aligned}D\psi = \gamma^1(V_1\psi - \frac{1}{2}\gamma^1\psi) + \gamma^2(V_2\psi - \frac{1}{2}\gamma^2\psi)&\\ = \gamma^a V_a\psi + \psi.&\end{aligned} \tag{9.12}$$

The last formula is manifestly covariant under rotations and so holds at any point on S^2. It can be written

$$D = \gamma^a V_a + 1. \tag{9.13}$$

Some different derivations of this formula can be found in [10] (for their case $k = 0$) and in [3].

The expression (9.13) is equal to $-2S \cdot V + 1$ using the notation $S \cdot V = S_a V_b \delta^{ab}$. This can be written in terms of the Casimir operators for SU(2) representations $c_1 = -S^2 = -S \cdot S$ on $\mathbb{C}^2$, $c_2 = -V^2$ on $L^2(S^2)$, and 'total angular momentum' $c = -(V + S)^2$ on the product $\mathcal{H}$, as

$$D = c - c_1 - c_2 + 1. \tag{9.14}$$

These Casimir operators all commute with each other.

The eigenvalues of these Casimir operators are as follows. The operator $c_1 = \frac{1}{2}(\frac{1}{2} + 1) = \frac{3}{4}$ on $\mathbb{C}^2$. The Casimir $c_2 = k(k + 1)$ on $L^2(S^2)$, for $k = 0, 1, 2, \ldots$, each an irreducible representation of spherical harmonics. The decomposition for the total spin representation $V + S$ on $\mathbb{C}^2$ tensored with the spin k subspace of $L^2(S^2)$ is $\frac{1}{2} \otimes k = (k - \frac{1}{2}) \oplus (k + \frac{1}{2})$, so that $c = j(j + 1)$ for $j = k \pm \frac{1}{2}$. This gives eigenvalues of D as

$$\lambda = j(j + 1) - k(k + 1) + \frac{1}{4} = \pm(j + \frac{1}{2}) \tag{9.15}$$

for $j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$. The irreducible representations of SU(2) are labelled with spin j and the sign $\pm$, and so each eigenvalue of D has multiplicity $2j+1$.

A simple check for the spectrum is to compare the eigenvalues of D^2 to the Casimir operator c. According to Parthasarathy's formula [1, Th 3.1], $D^2 = c + R/8$, with $R = 2$ the scalar curvature of the sphere. This equation is verified since $(j+\frac{1}{2})^2 = j(j+1) + \frac{2}{8}$.

9.2 Spectral triple for fuzzy sphere

The Dirac operator for the fuzzy sphere is easy to describe. The Hilbert space is $\mathcal{H} = \mathbb{C}^2 \otimes M_N(\mathbb{C})$ and the algebra $\mathcal{A} = M_N(\mathbb{C})$, acting by matrix multiplication on the left and right. The Grosse-Presnajder Dirac operator [10] is

$$D = \gamma^a \otimes [X_a, \cdot\,] + I \tag{9.16}$$

with the $(0,3)$ gamma matrices γ^a, and X_a the generators of the Lie algebra of SU(2) in the $N \times N$ matrix representation.

Exercise 14. Show that the Dirac operator (9.16) is Hermitian and satisfies the first-order condition.

There is a real structure (see Ch 12) and the trivial chirality operator $\gamma = I$, making this a spectral triple with KO-dimension 3.

λ	$1-N$	$\dots$	-2	-1	1	2	$\dots$	$N-1$	N
j	$N-\frac{3}{2}$	$\dots$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\dots$	$N-\frac{3}{2}$	$N-\frac{1}{2}$
m	$2N-2$	$\dots$	4	2	2	4	$\dots$	$2N-2$	$2N$

Figure 9.1: Eigenvalues λ of the Dirac operator for the fuzzy sphere and their spin j and multiplicity m.

There is an obvious analogy to the commutative Dirac operator on S^2 by comparing with (9.13), replacing the derivatives V_a with the commutators $\rho(X_a) = [X_a, \cdot\,]$. Since the latter play exactly the same role, as a representation of the Lie algebra, the operator D can be written in terms of Casimirs exactly as in (9.14) by just replacing V_a with $\rho(X_a)$. The result for the eigenvalues of D is the same as (9.15), but with the restriction $k \leq N-1$ from (2.10).

As a result, the eigenvalues of D are asymmetric about 0. Each representation appears twice except the largest one, which arises from $\frac{1}{2}\otimes(N-1) = (N-\frac{3}{2})\oplus(N-\frac{1}{2})$. The representation $N-\frac{1}{2}$ comes with the $+$ sign in $j = k \pm \frac{1}{2}$, but there is no copy of it with the $-$ sign. The eigenvalues are given in Figure 9.1.

Exercise 15. List the eigenvalues and their multiplicities for the Dirac operator of the fuzzy sphere with $N = 4$ and check that the multiplicities add up to the dimension of

$\mathcal{H}$. Why is it impossible to have a chirality operator γ satisfying $D\gamma = -\gamma D$?

The relation to the spectrum of the commutative sphere is simple. Taking $N \to \infty$ in Figure 9.1 gives the spectrum of the Dirac operator on S^2.

Chapter 10

The fuzzy torus

The fuzzy torus is another example of a fuzzy space with symmetry, but it is somewhat different to the fuzzy sphere. The symmetry is a finite group, depending on the matrix size, rather than a Lie group. The eigenvalues of the Dirac operator differ from the commutative ones, but converge to them as the matrices increase in size.

10.1 Matrix generators

The fuzzy torus is a choice of two unitary matrices $U, V \in M_N(\mathbb{C})$, for some N, such that

$$UV = qVU \tag{10.1}$$

for some complex number q, together with a differential operator that specifies the geometry of the torus.

Exercise 16. If ξ is an eigenvector of U with eigenvalue λ, compute the eigenvalue of the vector $V\xi$. Why does this imply that q is a root of 1?

It is possible to study (10.1) when q is not a root of 1 but then U and V have to be operators on an infinite-dimensional Hilbert space. This is called the irrational noncommutative torus, and the geometry is rather different.

If $q = 1$, then the matrices U and V can be diagonalised simultaneously, with eigenvalues $e^{i\theta}$ and $e^{i\phi}$, defining N^2 points on the torus $S^1 \times S^1$.

However, in the non-commutative case, only one of the operators can be diagonalised. The standard example of such matrices are the clock and shift operators

$$C = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \mathfrak{q} & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \mathfrak{q}^{N-1} \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 & \dots & 1 \\ 1 & 0 & \dots & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{10.2}$$

which satisfy $CS = \mathfrak{q}SC$, with $\mathfrak{q}^N = 1$. The *square fuzzy torus* is obtained by putting

$$U = C, \quad V = S, \quad q = \mathfrak{q}, \tag{10.3}$$

providing an example of the torus relation (10.1). Other examples can be constructed by defining U and V to be monomials in C and S, but this is not explored here.

10.2 Laplace operator

A Laplace operator can be defined by [14]

$$\Delta = \frac{-1}{(q^{1/2} - q^{-1/2})^2} \left([U,[U^*,\cdot\,]] + [V,[V^*,\cdot\,]] \right), \quad (10.4)$$

This formula is a bit different to (1.7) because U and V are not anti-Hermitian. This can be fixed by taking the linear combinations

$$\begin{aligned} K_1 &= -\frac{1}{2}(U + U^*), \qquad & K_2 &= -\frac{i}{2}(U^* - U), \\ K_3 &= \frac{1}{2}(V + V^*), \qquad & K_4 &= \frac{i}{2}(V^* - V). \end{aligned} \quad (10.5)$$

These are Hermitian, but rescaling them as

$$X_i = \frac{1}{q^{1/2} - q^{-1/2}} K_i \quad (10.6)$$

gives a set of anti-Hermitian operators. It is straightforward to check that the Laplacian is then equal to (1.7), with four terms.

The eigenvalues of Δ can be computed in terms of the *quantum integers*. Given an integer n and a square root $q^{1/2}$, quantum n is defined as

$$[n]_q = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}. \tag{10.7}$$

The quantum integers appear in a number of mathematical fields, particularly where derivatives are replaced by finite differences. The main feature is that $[n]_q \to n$ as $q \to 1$. A simple case is the result of the following exercise.

Exercise 17. For the square fuzzy torus (10.3), show that the eigenvectors of Δ are monomials in C and S, and that the eigenvalues are

$$\lambda_{k,l} = [k]_q^2 + [l]_q^2 \tag{10.8}$$

for $k, l = 1, 2, \ldots, N$.

The eigenvalues in the exercise can be compared with those of the Laplacian $-\frac{\partial^2}{\partial\theta^2} - \frac{\partial^2}{\partial\phi^2}$ on the Riemannian square torus. This has eigenvalues $k^2 + l^2$ for all integers k, l. So, the fuzzy torus spectrum is a deformed version of a finite subset of the commutative torus spectrum.

This can be examined in more detail by looking at the case $q = e^{2\pi i/N}$. Then

$$[n]_q = \frac{\sin \pi n/N}{\sin \pi/N} \tag{10.9}$$

which is closest to n when n is close to 0 mod N. In other words, the low-energy modes of the fuzzy space behave like the commutative ones.

Despite the fact that U and V do not commute, there is still an action of the finite group $\mathbb{Z}_N \times \mathbb{Z}_N$ on $M_N(\mathbb{C})$ (as usual $\mathbb{Z}_N$ means $\mathbb{Z}/N\mathbb{Z}$), assuming that $q^N = 1$. This is

$$(j, m) \colon a \mapsto V^{-j} U^m a U^{-m} V^j. \tag{10.10}$$

This group action commutes with Δ. In this sense, the fuzzy torus retains a similarity to a finite lattice of points on a (commutative) torus.

Exercise 18. Show that the eigenvectors obtained in Exercise 17 are also eigenvectors for generators of $\mathbb{Z}_N \times \mathbb{Z}_N$, and compute their eigenvalues.

10.3 Spectral triple

A spectral triple is constructed in the following way [4]. The $*$-algebra is $\mathcal{A} = M_N(\mathbb{C})$. The Hilbert space is $\mathcal{H} = \mathbb{C}^4 \otimes M_N(\mathbb{C})$. The space $\mathbb{C}^4$ is the spinor space with type $(0, 4)$ gamma matrices acting on it. This has a chirality operator and a real structure which are extended to $\mathcal{H}$ as $\gamma \otimes 1$ and $J \otimes *$. As a consequence, the KO-dimension of the spectral triple will be $s = 4$.

A fourth root of q is chosen, $q^{1/4}$. The Dirac operator is

$$D = \frac{1}{2(q^{1/4} - q^{-1/4})} \sum_i \gamma^i \otimes [K_i, \cdot\,] + \frac{1}{2(q^{1/4} + q^{-1/4})} \sum_{i<j<k} \gamma^i \gamma^j \gamma^k \otimes \{K_{ijk}, \cdot\} \quad (10.11)$$

with $K_{234} = K_1$, $K_{134} = -K_2$, $K_{124} = -K_3$, $K_{123} = K_4$.

The commutator terms are the fuzzy analogue of derivatives, according to Ch 3, and the imaginary prefactor times K is an anti-Hermitian matrix, as required.

The anticommutator terms are the fuzzy analogues of the spin connection coefficients, with a real prefactor times K being a Hermitian matrix. These terms become the multiplication by functions (and gamma matrices) in the commutative limit, according to Ch 3.

The exact prefactors are rather delicate and cannot be easily predicted by a commutative limit. They are designed to give a reasonable spectrum for the Dirac operator. For the square fuzzy torus, the eigenvalues of D are

$$\lambda_{k,l,\pm} = \pm\sqrt{\left[k + \frac{1}{2}\right]_q^2 + \left[l + \frac{1}{2}\right]_q^2}. \quad (10.12)$$

This is a long calculation, broken down into stages in [4]. The positive eigenvalues are shown in Figure 10.1. These behave like the commutative Dirac operator near the origin (a 'Dirac cone') but soften at large k, l, to give a periodic graph, somewhat like a plot of the dispersion relation for a lattice in solid state physics.

The squares of these eigenvalues are almost the same as the eigenvalues of the Laplacian, (10.8). The difference is the shift $+\frac{1}{2}$ in the values of k and l. In the commutative case, there are four different spin structures for the torus, depending on whether one chooses periodic or anti-periodic boundary conditions for the identification of the trivial spinor bundle on $\mathbb{R}^2$ to give a spinor bundle on $S^1 \times S^1$. For the anti-periodic boundary conditions along both circles, the Fourier modes become $e^{i(k+1/2)\theta}$ and $e^{i(l+1/2)\phi}$, which leads to this shift in the eigenvalues. Thus the fuzzy torus spectral triple has the fuzzy analogue of the anti-periodic boundary conditions in both circles.

It is possible to give fuzzy versions of all four spin structures using a different choice of U and V and a slightly more involved construction [4].

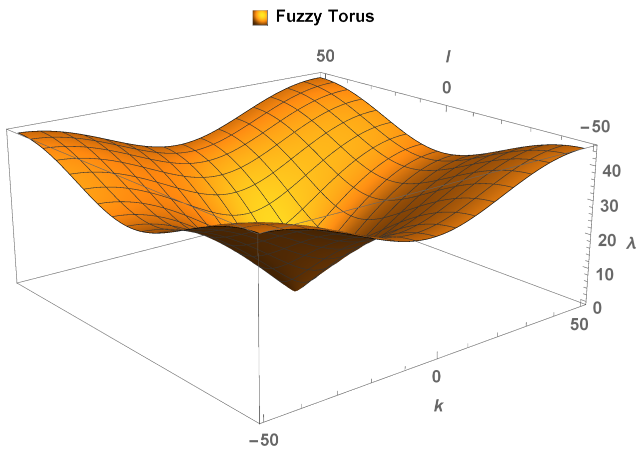

Figure 10.1: The positive eigenvalues of D for the square fuzzy torus with $N = 100$, plotted as a continuous surface.

Chapter 11

Fluctuations

Two component parts of the standard model have been described so far: the space-time manifold, at least with a Euclidean metric (Chapters 4, 5, 6), and the internal space (Chapters 7, 8).

Following the idea of Kaluza-Klein, one should take the product of the two to get a 4-dimensional space-time metric together with some bosonic matter fields. This product geometry is called the *vacuum* geometry because all of the gauge fields are zero and the Higgs field is constant over space-time.

More general matter field configurations can be obtained from the vacuum geometry by deforming it, to arrive at what one calls fluctuations of the vacuum. This results in non-trivial gauge fields and a Higgs field that can vary from point to point. It's a wonderful fact that all of the known bosonic fields in nature arise this way.

11.1 Products

Let $(s_1, \mathcal{A}_1, \mathcal{H}_1, D_1, J_1)$ and $(s_2, \mathcal{A}_2, \mathcal{H}_2, D_2, J_2)$ be two real spectral triples with s_1 and s_2 even. Then there is a product spectral triple constructed in the following way.

$$\begin{aligned}
\mathcal{A} &= \mathcal{A}_1 \otimes \mathcal{A}_2 \\
\mathcal{H} &= \mathcal{H}_1 \otimes \mathcal{H}_2 \\
l(a) &= l_1(a_1) \otimes l_2(a_2) \\
\gamma &= \gamma_1 \otimes \gamma_2 \\
D &= D_1 \otimes I + \gamma_1 \otimes D_2
\end{aligned} \tag{11.1}$$

The formula for D is slightly surprising, but it is similar to the rule for products of gamma matrices. Obviously one gets a different formula for the Dirac operator by exchanging the two spectral triples, but the two formulas are unitarily equivalent.

The following exercise explains some properties of D.

Exercise 19. Check that D anticommutes with γ. Show that $D^2 = D_1^2 \otimes I + I \otimes D_2^2$.

Note that the formula for D^2 is what one would expect for a Laplace operator on a product space.

The real structure on the product is

$$\begin{aligned} s &= s_1 + s_2 \qquad \mod 8 \\ J &= \begin{cases} J_1 \otimes J_2 & \text{if } s_1 = 0, 4 \\ J_1 \otimes J_2 \gamma_2 & \text{if } s_1 = 2, 6 \end{cases} \end{aligned} \tag{11.2}$$

Exercise 20. For each even s_1, s_2, check that the real structure has KO-dimension $s_1 + s_2$ according to the table of signs, Figure 5.1. What is wrong with the definition if the two formulas for J in (11.2) are interchanged?

There is also a product if s_2 is odd, which is explained (along with the even cases) in [16]. The product of spectral triples is directly analogous to the product of gamma matrices, which is explained in [3].

11.2 Standard model vacuum

The standard model geometry is obtained from the product of the $s_1 = 4$ spectral triple for the Euclidean spacetime M with the $s_2 = 6$ internal space spectral triple, which is labelled F. The result is a spectral triple with

$s = 6 + 4 = 2$, and

$$\begin{aligned}
\mathcal{A} &= \mathcal{A}_M \otimes \mathcal{A}_F = C^\infty(M, \mathcal{A}_F) \\
\mathcal{H} &= \mathcal{H}_M \otimes \mathcal{H}_F = L^2(M, S \otimes \mathcal{H}_F) \\
D_0 &= D_M \otimes I + \gamma_M \otimes D_F \\
J &= J_M \otimes J_F.
\end{aligned} \tag{11.3}$$

The elements of the algebra are now matrices $a(x)$ that depend on $x \in M$.

The elements of the Hilbert space (for one generation) are the block off-diagonal matrices $\Psi(x)$ of (8.1), whose matrix components are now Dirac spinor fields on M. This is too many spinor components because the field $\nu_L(x)$ should be a left-handed Weyl spinor, not a Dirac spinor.

This problem is fixed by defining the chiral subspace $\mathcal{H}^+ \subset \mathcal{H}$ by

$$\gamma\Psi = \Psi, \tag{11.4}$$

Since $\gamma = \gamma_M \otimes \gamma_F$, the eigenvalues of γ_M and γ_F must be equal for vectors in $\mathcal{H}^+$. In this subspace, the field $\nu_L(x)$ is correctly a left-handed spinor and $\nu_R(x)$ a right-handed spinor, and so the physical fields in quantum field theory lie in $\mathcal{H}^+$. However, one cannot just replace $\mathcal{H}$ with $\mathcal{H}^+$ in the spectral triple because a Dirac operator will map $\mathcal{H}^+$ to its orthogonal complement, $\mathcal{H}^-$.

The space $\mathcal{H}^+$ still contains both fields and their conjugates as independent vectors. This is not a problem because in quantum field theory, a functional integral over a complex field is obtained by integrating over both the field and its conjugate as independent variables (in the same way that integrating over the complex plane is written $\int \mathrm{d}z\mathrm{d}\overline{z}$).

11.3 Gauge fields

The Dirac operator in the product space (11.3) is denoted D_0 and is called the vacuum geometry. The spectral triples of interest in physics are more general than D_0, just as geometries on a product space are more general than the product of geometries on the two spaces. As noted earlier (Chapter 7), a gauge field occurs in Kaluza-Klein theory as the off-diagonal components of the metric tensor on the product space.

A similar phenomenon happens here. Let $\Omega(\mathcal{A}, D_0) = \{\omega\}$ be the set of operators that are finite sums

$$\omega = \sum_i l(a_i)[D_0, l(b_i)] \tag{11.5}$$

for a_i, $b_i \in \mathcal{A}$. Then a self-adjoint element $\omega = \omega^*$ determines an *internal fluctuation* (or *inner fluctuation*) of D_0,

$$D = D_0 + \omega + J\omega J^{-1}. \tag{11.6}$$

Note that ω commutes with the right action of $\mathcal{A}$, whereas $J\omega J^{-1}$ commutes with the left action. Therefore D satisfies the first-order condition. Similarly, one can check the other axioms for the Dirac operator.

A gauge transformation is determined by a unitary element of $u \in \mathcal{A}$, in the same way as (7.5),

$$\rho(u) = l(u)r(u^{-1}). \tag{11.7}$$

The element u has to satisfy the unimodularity condition

$$\det{}_{\mathcal{H}_G} l(u(x)) = 1 \tag{11.8}$$

for each $x \in M$, taking the determinant over a one-generation internal Hilbert space.

For any unitary operator U, D is transformed into

$$D' = UDU^{-1} = D + U[D, U^{-1}]. \tag{11.9}$$

Applying this to $U = \rho(u)$ gives

$$D' - D = \rho(u)[D, \rho(u)^{-1}] = l(u)[D, l(u^*)] + r(u^*)[D, r(u)] \tag{11.10}$$

Using (11.6) results in $D' = D_0 + \omega' + J\omega' J^{-1}$ with

$$\begin{aligned} \omega' =& l(u)[D, l(u^*)] + \omega \\ &= l(u)[D_0, l(u^*)] + l(u)[\omega, l(u^*)] + \omega \\ &= l(u)[D_0, l(u^*)] + l(u)\,\omega\, l(u^*). \end{aligned} \tag{11.11}$$

Note that $\omega' \in \Omega(\mathcal{A}, D_0)$ and is self-adjoint. Therefore D' is also an internal fluctuation of D_0. The formula (11.11) is an analogue of the usual formula for the gauge transformation of a gauge potential, but it is more general because D_0 contains matrices of the internal space as well as derivatives.

To understand the formalism in terms of the usual fields, the internal fluctuations (11.5) are split into two parts according to the two parts of D_0 in (11.3).

$$\begin{aligned}\omega &= \omega_M + \omega_F \\ &= \sum_i l(a_i)[D_M \otimes I, l(b_i)] + \sum_i l(a_i)[\gamma_M \otimes D_F, l(b_i)]\end{aligned} \tag{11.12}$$

Using the local formula (9.3) for the Dirac operator, the first term is

$$\sum_i l(a_i)[D_M \otimes I, l(b_i)] = \sum_i l(a_i)\gamma^c \nabla_{e_c} l(b_i) \tag{11.13}$$

which is a matrix-valued one-form on M, contracted with gamma matrices (a *slashed one-form*). The second term involves the commutator of finite-dimensional matrices and does not have any derivatives or gamma matrices (apart from the chirality operator). This term is therefore a matrix-valued function of x which maps left-handed particle types to right-handed ones.

The gauge transformations also split,

$$\begin{aligned}\omega' = l(u)[D_M \otimes I, l(u^*)] + l(u)[\gamma_M \otimes D_F, l(u^*)] \\ + l(u)\,\omega_M\,l(u^*) + l(u)\,\omega_F\,l(u^*)\end{aligned} \qquad (11.14)$$

so one can regard each term as transforming separately. These are

$$\omega'_M = l(u)[D_M \otimes I, l(u^*)] + l(u)\,\omega_M\,l(u^*) \qquad (11.15)$$

which is the standard gauge transformation formula for a gauge potential, and

$$\begin{aligned}\omega'_F &= l(u)[\gamma_M \otimes D_F, l(u^*)] + l(u)\,\omega_F\,l(u^*) \\ &= l(u)(\gamma_M \otimes D_F + \omega_F)l(u^*) - \gamma_M \otimes D_F\end{aligned} \qquad (11.16)$$

which can be written

$$\omega'_F + \gamma_M \otimes D_R = l(u)(\gamma_M \otimes D_R + \omega_F)l(u^*) \qquad (11.17)$$

using the decomposition (8.10) for D_F.

Therefore one can interpret ω_M as a gauge field, and for the standard model one recovers the usual $U(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$ gauge field as the subspace of fluctuations that satisfy

$$\mathrm{Tr}_{\mathcal{H}_G}\,\omega_M(x) = 0 \qquad (11.18)$$

for each $x \in M$. This condition is a counterpart of the unimodularity condition for the gauge group, and is preserved by unimodular gauge transformations.

The term $\gamma_M \otimes D_R + \omega_F$ is interpreted as a matrix containing coupling constants and the Higgs field. Recall that D_F contains a matrix that determine the masses of the fermions. These matrix elements are not all constant under gauge transformations, and the fluctuations in ω_F that vary with x can be parameterised by Higgs fields. For the standard model, the lepton and quark mass terms transform non-trivially under the $\mathrm{U}(1) \times \mathrm{SU}(2)$ part of the gauge group and this is reproduced here.

Chapter 12

Matrix spectral triples

Chapter 13

Quantum gravity and matter

Chapter 14

Random Dirac operators

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