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Linearised Regge Calculus

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ABSTRACT Linearised Regge Calculus is developed in some detail, for linearisations around flat space with a positive metric. The linearised metric and curvature are defined as distributions, and the differential operators relating them and other geometric quantities are described. The spaces of distributions which arise for a given triangulation are related to some chain groups in simplicial topology. The calculation of curvature from the metric is transcribed by this method into some sequences of chain groups, with a modified form of chain boundary operator. The homology of these sequences is calculated using an extension of the usual duality theorem for the homology and cohomology of a simplicial manifold. These results allow a proof of the 'fundamental theorem' of linearised Regge calculus.

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1. Introduction

Regge Calculus was introduced as an approximation to General Relativity (Regge 1961). Because the approximation has a geometry in its own right (Regge 1961, Barrett 1987a), the relations between the approximating quantities (metric, connection, curvature, etc.) have a very similar structure to the relations between the usual smooth geometric quantities. This paper examines the situation for linearisations around flat space with a Euclidean signature metric. Section 2 summarises the linearised theory for general relativity. Cartan's method of 'moving frames' is exploited as this suits the development of Regge's theory. The analogous geometric quantities in linearised Regge calculus are introduced in §3. They are particular distributions related in a simple way to the simplicial structure of the triangulation. The equations satisfied are (with some minor modifications) the same equations as in linearised general relativity, extended to distributions. The machinery introduced in §3 allows a proof of the 'fundamental theorem' of linearised Regge calculus (Barrett 1987b) to be given. The proof is given in §3, but depends on the results of §5. This theorem, and the general theory of §3 and §4 explains the starting point of a paper (Barrett 1988) concerning the convergence problem for Regge calculus.

Section 4 explains why the theory introduced in §3 is the linearisation of the original Regge calculus, and hence justifies the naming of the spaces introduced in an ad hoc way in §3.

In §5, the homology of the sequences of chain groups introduced is calculated. Due to the tensorial nature of the distributions, it is necessary to introduce an unusual type of block dissection to reduce the homology to standard simplicial homology. The manner in which this is done is a modification of a standard method of proof of Lefschetz duality for triangulated manifolds.

2. Linearised General Relativity

Cartan used the exterior calculus to expose the structure of calculations in Riemannian geometry. He introduced a basis set of 1-forms θ^a which have a fixed normalisation, and showed that one can calculate the curvature 2-form components ${\rho_a}^b$ from the equations

$$\mathrm{d}\theta^a = -{\Omega_b}^a \wedge \theta^b \tag{1}$$

$${\rho_a}^c = \mathrm{d}{\Omega_a}^c - {\Omega_a}^b \wedge {\Omega_b}^c. \tag{2}$$

The connection components ${\Omega_b}^a$ can be calculated from (1), and then these in turn determine the curvature from (2). For a desciption of this theory, we refer the reader to Penrose and Rindler (1984), whose definitions and conventions we follow.

In the linearised theory (Sachs and Bergmann 1958, Penrose and Rindler 1984) we imagine a smooth one-parameter family of metrics [1], with the flat space metric $(\mathbf{R}^4, g)$ at the value 0 of the parameter u. For the 1-forms θ^a we imagine a smooth one-parameter family with a constant normalisation independent of u. Further, we suppose that at $u = 0$ the $\{\theta^a\}$ form a holonomic basis, and the normalisation is such that $e^a = \theta^a(0)$ is the unit vector valued form (i.e. the identity mapping from $\mathbf{R}^4$ into itself). (We identify the tangent space of $\mathbf{R}^4$ with $\mathbf{R}^4$ itself.) Because of this choice of e^a there is no distinction in the linearised theory between the 'frame indicies' $a, b, c\ldots$ expressed explicitly and the suppressed tangent space indicies of the exterior calculus.

Clearly ${\Omega_a}^b(0) = {\rho_a}^b(0) = 0$. If the derivatives of the following quantities at $u = 0$ are defined

$$\epsilon^a = \frac{\mathrm{d}\theta^a}{\mathrm{d}u}(0)$$

$${\omega_a}^b = \frac{\mathrm{d}{\Omega_a}^b}{\mathrm{d}u}(0)$$

$${R_a}^b = \frac{\mathrm{d}{\rho_a}^b}{\mathrm{d}u}(0)$$

then equations (1) and (2) imply

$$\mathrm{d}\epsilon^a = -{\omega_b}^a \wedge e^b \tag{3}$$

$${R_a}^b = \mathrm{d}{\omega_a}^b. \tag{4}$$

We take equations (3) and (4) to define the linearised theory, and call any quantities satisfying these equations a linearised tetrad (ϵ), linearised connection (ω) and linearised curvature (R). We shall not address the question of the existence of a one-parameter family of space-times with these the derivatives at parameter 0. Writing in the suppressed indicies on the tetrads, the tensor

$$h_{cd} = g_{ab}(e_c{}^a \epsilon_d{}^b + \epsilon_c{}^a e_d{}^b) = \epsilon_{dc} + \epsilon_{cd} \tag{5}$$

can be shown to be the linearised metric, i.e. the derivative of the one- parameter family of metrics at $u = 0$.

Examination of equations (3) and (4) shows that, for fields on a topologically trivial open region U of $\mathbf{R}^4$, any 2-form $R_a{}^b$ which satisfies the (linearised) Bianchi identity

$$\mathrm{d}R_a{}^b = 0 \tag{6}$$

comes from a linearised metric; one can integrate (3) and (4) if the de Rham cohomology group $H^2(U;\mathbf{R})$ is zero [2].

There are similar considerations for the 'gauge freedom'. If $H^1(U;\mathbf{R})$ vanishes, one can see that if any two (linearised) tetrads ϵ and ϵ' have the same curvature, then they differ only by a 1-form of the form

$$-\Lambda_b{}^a e^b + \mathrm{d}\alpha^a \tag{7}$$

where Λ is an antisymmetric tensor when the upper index is lowered, and consequently

$$h'_{ab} - h_{ab} = \nabla_b \alpha_a + \nabla_a \alpha_b \tag{8}$$

The two metrics are related by a linearised diffeomorphism.

All the above information can be codified in a diagram. The space of smooth k-forms on U with values in a vector space V is denoted $c^k(U;V)$. The antisymmetric tensor product of vector spaces is denoted by $\wedge$, and the symmetric tensor product by $\odot$.

$$\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \uparrow & & & & & & \\
 & & c^0(U;\mathbf{R}^4\odot\mathbf{R}^4) & & 0 & & & & \\
 & & \uparrow S & & \uparrow & & & & \\
c^0(U;\mathbf{R}^4) & \stackrel{\mathrm{d}}{\rightarrow} & c^1(U;\mathbf{R}^4) & \stackrel{\mathrm{d}}{\rightarrow} & c^2(U;\mathbf{R}^4) & \stackrel{\mathrm{d}}{\rightarrow} & c^3(U;\mathbf{R}^4) & & \\
 & & \uparrow W & & \uparrow W & & \uparrow W & & \\
 & & c^0(U;\mathbf{R}^4\wedge\mathbf{R}^4) & \stackrel{\mathrm{d}}{\rightarrow} & c^1(U;\mathbf{R}^4\wedge\mathbf{R}^4) & \stackrel{\mathrm{d}}{\rightarrow} & c^2(U;\mathbf{R}^4\wedge\mathbf{R}^4) & \stackrel{\mathrm{d}}{\rightarrow} & c^3(U;\mathbf{R}^4\wedge\mathbf{R}^4) \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{9}$$

S is the symmetrisation map of equation (5), and W is the map

$$W\colon\ x_b{}^a \longrightarrow -x_b{}^a \wedge e^b \tag{10}$$

which increases the degree of the form x by one.

The curvature lives in $c^2(U;\mathbf{R}^4 \wedge \mathbf{R}^4)$, and lies in the kernel of d and W. $W(R) = 0$ expresses the special symmetry properties of a Riemann curvature tensor r_{abcd} which are not already assured by the differential form notation, namely $r_{[abc]d} = 0$.

Proposition The diagram is a commutative diagram of linear maps. The vertical columns are exact and the two horizontal ones have homology $H^*(U;\mathbf{R})$ multiplied by coefficients in the appropriate linear space. The mapping $\Phi = \mathrm{d}W^{-1}\mathrm{d}S^{-1}\colon c^0(U;\mathbf{R}^4\odot\mathbf{R}^4) \rightarrow c^2(U;\mathbf{R}^4\wedge\mathbf{R}^4)$ is well-defined and its image obeys $\mathrm{Im}\,(\Phi) \subseteq \ker W \cap \ker \mathrm{d}$ with equality if $H^2(U) = 0$. If $H^1(U)$ vanishes, then $\ker\Phi = \mathrm{Im}\,(S\mathrm{d})$.

The map Φ is the main object of interest because it calculates curvatures from metrics.

Proof of the proposition: The calculation of curvature from a metric $h \in c^0(U; \mathbf{R}^4 \odot \mathbf{R}^4)$ can be traced through the diagram: since S is onto, one can pick a tetrad $S^{-1}h$ which is mapped onto h by S, and then one obtains a connection by applying the operator $W^{-1}d$, using the fact that W: $c^1(U; \mathbf{R}^4 \wedge \mathbf{R}^4) \to c^2(U; \mathbf{R}^4)$ is a 1–1 mapping (Penrose and Rindler 1984 §4.13). The different choices of $S^{-1}h$ result in connections differing by elements $\mathrm{d}\Lambda$, for $\Lambda \in c^0(U; \mathbf{R}^4 \wedge \mathbf{R}^4)$. Finally one applies the operator d to the connection, obtaining the curvature R, and eliminating the ambiguity, since $\mathrm{d}^2 = 0$. The properties $\mathrm{d}R = 0$ and $WR = 0$ follow from the commutativity of the diagram and using $\mathrm{d}^2 = 0$.

Similarly one can examine the problem discussed above of determining the metric given a curvature, and its 'gauge freedom', by tracing the mappings involved through the diagram and using the exactness properties. The details are left to the reader.□

3. Linearised Regge Calculus

Throughout this paper, we shall consider a particular fixed simplicial 4-manifold with boundary, consisting of the union of a finite number of simplexes[4] in $\mathbf{R}^4$. The manifold is denoted M, the boundary ∂M, and the underlying simplicial complex (of M) K, whilst L denotes the subcomplex which forms the boundary. M has a natural standard orientation: that given by the embedding in $\mathbf{R}^4$ acting as a coordinate system. We adopt the definitions and conventions of sign associated with orientation and integration of de Rham (1984).

The standard flat, positive, metric g in $\mathbf{R}^4$ defines a particular piecewise-flat metric for this triangulation of M. The space of metrics G is defined to be the space of piecewise-flat metrics for this triangulation (i.e. metrics which are flat on each simplex) which all have the same *boundary* metric, that induced by g. G is equal to the space of lengths of the interior edges. The tangent space to G at g is the space of linearised configurations of Regge calculus. Thus a linearised configuration consists of two parts, the "zeroth order metric" g, and a tangent vector, the "first order metric" h, in the same manner as for linearised general relativity. One can alternatively think of a linearised configuration as an equivalence class of one-parameter families of configurations, having the same derivatives at $g \in G$.

The first order metric h can be represented as a collection of symmetric bilinear forms; one on the tangent space to each simplex, and such that if $\boldsymbol{\varphi}$ is a face of simplex $\boldsymbol{\sigma}$, then the bilinear form on $\boldsymbol{\varphi}$ is the restriction of the form on $\boldsymbol{\sigma}$, and such that the bilinear form on any boundary simplex is zero. This will be discussed in more detail below (§3.5, §4).

3.1 Currents

The smooth fields of linearised general relativity are replaced by distributions in Regge Calculus. We need generalisations of the ordinary concept of distribution (Friedlander 1982), to cope with the distributional version of tensors and differential forms. The tensorial aspect is easily dealt with: our distributions are a map from test function space to a finite-dimensional vector space V, consisting of some appropriate tensor power of $\mathbf{R}^4$. This is equivalent to taking the tensor product of V with the space of distributions. One can say that each component is a distribution in the ordinary sense [5].

To cope with differential forms, we use de Rham's notion of a current (de Rham 1984). Although one could just regard a differential form as an antisymmetric tensor and use the above notion of tensor distribution, the algebra is simpler if one uses currents. The definition is as follows. A current of degree k (or k-current) is defined to be a continuous linear functional on the space of differential $(d-k)$-forms of compact support, in a d-dimensional manifold ($d=4$ here, but in some places d is left arbitrary for greater generality). Details of the topology are given by de Rham (1984). A smooth k-form α defines a current of degree k (also denoted α) by

$$\alpha[\beta] = \int \alpha \wedge \beta \tag{11}$$

where the integral is over $\mathbf{R}^d$ with the standard orientation. One defines the exterior derivative of the current T of degree k to be the $(k+1)$-current $\mathrm{d}T$:

$$\mathrm{d}T[\beta] = (-1)^{k+1} T[\mathrm{d}\beta] \tag{12}$$

which, partially integrating in formula (11), extends the exterior differentiation on smooth forms.

If T is a current of degree k and α a smooth j-form, one can define $\alpha \wedge T$ and $T \wedge \alpha$ by

$$\begin{aligned}(T \wedge \alpha)[\beta] &= T[\alpha \wedge \beta] \\ (\alpha \wedge T)[\beta] &= (-1)^{kj} T[\alpha \wedge \beta]\end{aligned} \tag{13}$$

3.2 Chain groups

An oriented k-simplex σ defines a current of degree $d-k$ by the formula

$$\sigma[\beta] = \int_\sigma \beta \tag{14}$$

A k-chain is a formal linear combination of a finite number of oriented k-simplexes, with real coeficients, i.e. $\sum \lambda_i \sigma_i$, $\lambda_i \in \mathbf{R}$. The k-th simplicial chain group we denote $C_k(K)$, real coefficients being understood unless otherwise stated. In most of the elementary definitions and results in simplicial topology we follow Maunder (1970).

Clearly the chain group is represented faithfully in the space of currents of degree $d-k$, by taking linear combinations of formula (14). We shall generally not make a distinction between an abstract chain and the current it defines in this way. Using Stokes' theorem in (14), one can see that the boundary operator ∂: $C_k(K) \rightarrow C_{k-1}(K)$ is related to the operator d acting on σ considered as a current by

$$\partial\sigma[\beta] = \sigma[\mathrm{d}\beta] = (-1)^{d-k+1} \mathrm{d}\sigma[\beta]$$

and so $\partial = (-1)^{d-k+1}\mathrm{d}$.

3.3 Cochain groups

The k-th cochain group of K, $C^k(K)$ is defined to be the dual vector space to $C_k(K)$, and the (co)boundary operator δ: $C^k(K) \to C^{k+1}(K)$ the adjoint of d. In this section, it is described how the relative cochain groups $C^k(K,L)$ can be represented as currents. It is necessary to introduce the spaces $\lambda^q(\boldsymbol{\sigma})$, the space of *constant q-forms on a simplex $\boldsymbol{\sigma}$*, and $\Lambda^q(\boldsymbol{\sigma})$, the space of *skew-linear q-forms on a simplex $\boldsymbol{\sigma}$*. These are defined as follows. We suppose $\boldsymbol{\sigma}$ has dimension n. Since $\boldsymbol{\sigma}$ is a subset of $\mathbf{R}^n$, a constant differential form on $\boldsymbol{\sigma}$ can be defined as one whose components in the standard coordinates $(x^1, \dots, x^n)$ are constants. A skew-linear q-form α is defined to be a differential form on $\boldsymbol{\sigma}$ such that its value at point x is related to its value at point x' by

$$\alpha_x(u_1, \dots u_q) - \alpha_{x'}(u_1, \dots u_q) = \beta(x - x', u_1, \dots, u_q) \tag{15}$$

for any q vectors $u_1,\dots\ u_q$, and a constant $(q+1)$-form β. Note that $\beta = \mathrm{d}\alpha$.

The relations between currents and cochains is simple enough on a single simplex. If $K(\boldsymbol{\sigma})$ is the simplicial complex consisting of the faces of $\boldsymbol{\sigma}$ (including $\boldsymbol{\sigma}$ itself), then there is a linear map $\Phi: \Lambda^k(\boldsymbol{\sigma}) \to C^k(K(\boldsymbol{\sigma}))$, given by integrating the k-form on the k-faces of $\boldsymbol{\sigma}$.

Proposition. *Φ is a linear isomorphism.*

Proof: The dimension of both spaces is the same. Suppose $\alpha \in \Lambda^k(\boldsymbol{\sigma})$, and $\Phi(\alpha) = 0$. From (15), α restricted to a k-face is constant. Suppose v is a vertex of $\boldsymbol{\sigma}$, then α_v restricted to each k-face containing v is zero, which implies that $\alpha_v = 0$. Now α at a general point is just a linear combination of its values on the vertices, and so is zero everywhere. □

Now consider the (d-dimensional) complex (K, L). A cochain $\gamma \in C^k(K,L)$ defines a cochain in $C^k(K(\boldsymbol{\sigma}))$ for each d-simplex $\boldsymbol{\sigma}$, and hence using the above isomorphism Φ^{-1}, a skew-linear k-form α on $\boldsymbol{\sigma}$. This form can be extended to any point $x \in \mathbf{R}^d$ by use of (15), with $x' \in \boldsymbol{\sigma}$. The extended form on $\mathbf{R}^d$ is denoted $\alpha(\boldsymbol{\sigma})$. The d-simplexes in K can be considered oriented simplexes, all with the standard orientation of M, and hence currents of degree 0, by (14). The current taken by summing over all d-simplexes in K

$$\sum \sigma \wedge \alpha(\boldsymbol{\sigma}) \tag{16}$$

defines *the current corresponding to the cochain* γ. Again, a distinction shall not in general be made between the cochain and the current which represents it. This current will be denoted by the same symbol, γ in this case. Taking the exterior derivative,

$$\mathrm{d}\gamma = \mathrm{d}\left(\sum \sigma \wedge \alpha(\boldsymbol{\sigma})\right) = \sum \sigma \wedge \mathrm{d}\alpha(\boldsymbol{\sigma}) - \partial\sigma \wedge \alpha(\boldsymbol{\sigma}). \tag{17}$$

The second term is zero, for the following reason. The boundary $\partial\sigma$ can be represented as a sum of $(d-1)$-simplexes. In general, one can show that if a k-form β is zero when restricted to a simplex $\boldsymbol{\tau}$, then $\tau \wedge \beta = 0$. This is the situation for $(d-1)$-simplexes $\boldsymbol{\tau} \in L$ which occur in the second term of (17), since the cochain γ is relative to L. For $(d-1)$-simplexes $\boldsymbol{\tau}$ in the interior of K, they appear twice in the second term of (17), and picking one particular orientation for $\boldsymbol{\tau}$, in the form $\tau \wedge (\alpha(\boldsymbol{\sigma}) - \alpha(\boldsymbol{\sigma}'))$, where $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ are the two d-simplexes which contain $\boldsymbol{\tau}$. Again, the form $\alpha(\boldsymbol{\sigma}) - \alpha(\boldsymbol{\sigma}')$ is zero when restricted to the face $\boldsymbol{\tau}$. Thus

$$\mathrm{d}\gamma = \sum \sigma \wedge \mathrm{d}\alpha(\boldsymbol{\sigma}). \tag{18}$$

With equation (18) established, one can now demonstrate the relationship between d and the coboundary operator δ, namely that $\mathrm{d} = \delta$ on the elements of $C^k(K,L)$. This follows fairly directly from (18), and Stokes' theorem.

3.4 Modified chain groups

The chain groups introduced so far are not sufficient to express all the quantities in linearised Regge calculus. It is necessary to introduce a simple modification of the chain groups and the boundary operator. The vector space $C_{n,k}(K)$ is defined to be the subspace of currents of degree $d-k+n$ spanned by currents of the form $\{\gamma\wedge\alpha\}$ for $\gamma\in C_k(K)$ and $\alpha\in\lambda^n(M)$, the space of constant n-forms on M. Note that if σ is an oriented k-simplex, a generator of $C_k(K)$, then $\sigma\wedge\alpha$ depends only on the restriction of α to the (geometric) simplex $\boldsymbol{\sigma}$. One could alternatively regard $C_{n,k}(K)$ as the direct sum over (geometric) k-simplexes $\boldsymbol{\sigma}$ of the spaces $C_k(K(\boldsymbol{\sigma}))\otimes\lambda^n(\boldsymbol{\sigma})$. This is written $\bigoplus_{\boldsymbol{\sigma}}(C_k(K(\boldsymbol{\sigma}))\otimes\lambda^n(\boldsymbol{\sigma}))$.

The boundary operator d sends $\gamma\wedge\alpha$ to $(\mathrm{d}\gamma)\wedge\alpha$, and so the image of d lies in $C_{n,k-1}(K)$. One can compare its action to the usual homology boundary operator ∂ acting on $C_k(K)\otimes\lambda^n(M)$. Whereas in the later the coefficients are unchanged by ∂, in our modified complex, as a map $\bigoplus_{\boldsymbol{\sigma}}(C_k(K(\boldsymbol{\sigma}))\otimes\lambda^n(\boldsymbol{\sigma}))\to\bigoplus_{\boldsymbol{\tau}}(C_{k-1}(K(\boldsymbol{\tau}))\otimes\lambda^n(\boldsymbol{\tau}))$, the element $\sigma\otimes\alpha$ is mapped by d to a sum of terms of the form $\tau\otimes\alpha'$, where $\boldsymbol{\tau}$ is a face of $\boldsymbol{\sigma}$, and α' is the form α *restricted to the face* $\boldsymbol{\tau}$.

Beside the usual sequences of cohomology

$$0\to C^0(K,L)\xrightarrow{\mathrm{d}}C^1(K,L)\xrightarrow{\mathrm{d}}\cdots\xrightarrow{\mathrm{d}}C^d(K,L)\to 0 \tag{19}$$

and homology

$$0\to C_d(K)\xrightarrow{\mathrm{d}}C_{d-1}(K)\xrightarrow{\mathrm{d}}\cdots\xrightarrow{\mathrm{d}}C_0(K)\to 0 \tag{20}$$

the modified chain groups allow us to construct the following sequences

$$0\xrightarrow{\mathrm{d}}C^0(K,L)\xrightarrow{\mathrm{d}}\cdots\xrightarrow{\mathrm{d}}C^{n-1}(K,L)\xrightarrow{\mathrm{d}}C_{n,d}(K)\xrightarrow{\mathrm{d}}C_{n,d-1}(K)\xrightarrow{\mathrm{d}}\cdots\xrightarrow{\mathrm{d}}C_{n,n}(K)\xrightarrow{\mathrm{d}}0 \tag{21}$$

for each integer n, for $0\le n\le d$. The case $n=0$ is (20), and the case $n=d$ is (19). These new sequences 'interpolate' between the homology chain complex and the cohomology chain complex.

It is also necessary to introduce $B(\boldsymbol{\sigma})$, the space of symmetric bilinear forms on the tangent space to $\boldsymbol{\sigma}$ (we can identify in a standard way the tangent spaces of each point of $\boldsymbol{\sigma}$ considered as a manifold, and speak of the tangent space to $\boldsymbol{\sigma}$), and $B_k(K)$, which is defined to be a direct sum of vector spaces, one for each k-simplex in K:

$$B_k(K)=\bigoplus_{\boldsymbol{\sigma}}C_k(K(\boldsymbol{\sigma}))\otimes B(\boldsymbol{\sigma})$$

The relevant boundary operator $\partial'\colon B_k(K)\to B_{k-1}(K)$ is defined as follows:

$$\sigma\otimes b\mapsto\sum\tau\otimes b(\boldsymbol{\tau}),\quad\text{where }\partial\sigma=\sum\tau$$

and $b(\boldsymbol{\tau})$ is the bilinear form b restricted to simplex $\boldsymbol{\tau}$.

The operator ∂' differs from the others introduced in that it does not reduce to the operator d acting on a suitable space of currents.

3.5 The geometric quantities

Having introduced the notation, we can now state how the principal geometric quantities of linearised Regge calculus are represented. The justification in terms of the non-linear Regge calculus will be postponed to §4.

The linearised metric is a element h of $B_4(K)$ such that $\partial' h = 0$. This ensures that the bilinear forms agree on common faces, and are zero when resticted to the boundary, as mentioned above. The linearised tetrad ϵ is an element of $C_{1,4}(K) \otimes \mathbf{R}^4$ which defines a linearised metric by the map s, which is defined to be the map

$$s\colon C_{1,k}(K) \otimes \mathbf{R}^4 \to B_k(K)$$

which linearly extends

$$\sigma \wedge \alpha^a \mapsto \sigma \otimes i_*(\alpha_{ab} + \alpha_{ba}),$$

where σ is a simplex, $\alpha_b{}^a$ is the tensor corresponding to the vector-valued 1-form α^a, the vector index is lowered with the background metric g, and i_* is the map restricting the bilinear form to the simplex. (Although $\sigma \wedge \alpha$ does not determine the 1-form α uniquely, it does determine its restriction to the simplex uniquely.) The map s should be compared with the map S of §2. Thus a linearised tetrad is any element of $C_{1,4}(K) \otimes \mathbf{R}^4$ such that $\partial' s\epsilon = 0$.

The (linearised) connection ω is an element of $C_3(K) \otimes (\mathbf{R}^4 \wedge \mathbf{R}^4)$, and the curvature an element R of $C_2(K) \otimes (\mathbf{R}^4 \wedge \mathbf{R}^4)$, related to the connection by $R = \mathrm{d}\omega$. Likewise $\mathrm{d}R = 0$ is the simplicial analog of the Bianchi identity. The map

$$\begin{aligned} w\colon\; C_k(K) \otimes (\mathbf{R}^4 \wedge \mathbf{R}^4) &\to C_{1,k}(K) \otimes \mathbf{R}^4 \\ \sigma \otimes \Lambda_a{}^b &\mapsto -\Lambda_a{}^b(\sigma \wedge e^a) \end{aligned}$$

is the analog of W in (10). Again, $\mathrm{d}w = w\mathrm{d}$. The relation between the connection and the tetrad is given by the equation $\mathrm{d}\epsilon = w\omega$ (Compare with (3)). These relations, and the gauge freedoms, can be expressed in the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & B_4(K) & \xrightarrow{-\partial'} & B_3(K) & & & & \\
 & & \uparrow s & & \uparrow s & & & & \\
0 \to C^0(K,L) \otimes \mathbf{R}^4 & \xrightarrow{\mathrm{d}} & C_{1,4}(K) \otimes \mathbf{R}^4 & \xrightarrow{\mathrm{d}} & C_{1,3}(K) \otimes \mathbf{R}^4 & \xrightarrow{\mathrm{d}} & C_{1,2}(K) \otimes \mathbf{R}^4 & & \\
 & & \uparrow w & & \uparrow w & & \uparrow w & & \\
0 & \to & C_4(K) \otimes \mathbf{R}^4 \wedge \mathbf{R}^4 & \xrightarrow{\mathrm{d}} & C_3(K) \otimes \mathbf{R}^4 \wedge \mathbf{R}^4 & \xrightarrow{\mathrm{d}} & C_2(K) \otimes \mathbf{R}^4 \wedge \mathbf{R}^4 & \xrightarrow{\mathrm{d}} & C_1(K) \otimes \mathbf{R}^4 \wedge \mathbf{R}^4 \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{22}$$

In fact one can extend this diagram further, both vertically and horizontally. Most of the extra spaces are not used in Regge calculus and so we shall omit them. The extension of the bottom right hand corner will be discussed further below (§3.9).

The vertical sequences are exact, as we shall now show. The sequences are essentially many copies of the following sequences, which are written in the fully extended forms, i.e. the sequences one would have if diagram (22) were extended. The first vertical sequence, reading from left to right in (22), is simply

$$0 \to \mathbf{R}^4 \wedge \mathbf{R}^4 \to \mathbf{R}^4 \otimes \mathbf{R}^4 \to \mathbf{R}^4 \odot \mathbf{R}^4 \to 0 \tag{23}$$

Let $\mathbf{R}^4 = \mathbf{R}^1_\perp \oplus \mathbf{R}^3$ be a decomposition of $\mathbf{R}^4$ into orthogonal subspaces. The $\mathbf{R}^3$ is the subspace of $\mathbf{R}^4$ parallel to a particular 3-simplex. Then the second sequence is

$$0 \to \mathbf{R}^4 \wedge \mathbf{R}^4 \to \mathbf{R}^3 \otimes \mathbf{R}^4 \to \mathbf{R}^3 \odot \mathbf{R}^3 \to 0 \tag{24}$$

where the maps involved are the obvious injections and orthogonal projections. The third vertical sequence, if the diagram were extended fully, is

$$0 \to \mathbf{R}^2_\perp \wedge \mathbf{R}^2_\perp \to \mathbf{R}^4 \wedge \mathbf{R}^4 \to \mathbf{R}^2 \otimes \mathbf{R}^4 \to \mathbf{R}^2 \odot \mathbf{R}^2 \to 0 \tag{25}$$

with the decomposition $\mathbf{R}^4 = \mathbf{R}^2_\perp \oplus \mathbf{R}^2$ of $\mathbf{R}^4$ into subspaces parallel and perpendicular to a 2-simplex. The final sequence, of which only $\mathbf{R}^4 \wedge \mathbf{R}^4$ appears in (22), would be

$$0 \to \mathbf{R}^3_\perp \wedge \mathbf{R}^3_\perp \to \mathbf{R}^4 \wedge \mathbf{R}^4 \to \mathbf{R}^1 \otimes \mathbf{R}^4 \to \mathbf{R}^1 \odot \mathbf{R}^1 \to 0 \tag{26}$$

One can easily see that these sequences are exact.

3.6 Calculating curvature from the metric

Let us define the space *metrics* $= \ker \partial' \subset B_4(K)$, and the space *curvatures* $= \ker \mathrm{d} \cap \ker w \subset C_2(K) \otimes (\mathbf{R}^4 \wedge \mathbf{R}^4)$

Proposition The map $\varphi_2 = \mathrm{d}(w^{-1})\mathrm{d}(s^{-1})$: *metrics* $\to C_2(K) \otimes (\mathbf{R}^4 \wedge \mathbf{R}^4)$ is uniquely defined, and $\mathrm{Im}\,(\varphi_2) \subseteq$ *curvatures*.

Proof: Using the vertical exactness, there exists an element ϵ of $C_{1,4}(K) \otimes \mathbf{R}^4$ such that $s\epsilon = h$. Using the commutativity of the diagram, $s(\mathrm{d}\epsilon) = 0$, and so there exists a unique element ω of $C_3(K) \otimes \mathbf{R}^4 \wedge \mathbf{R}^4$ such that $w\omega = \mathrm{d}\epsilon$. A second element ϵ' also satisfying $s\epsilon' = h$ differs from ϵ by an element $\Lambda \in C_4(K) \otimes \mathbf{R}^4 \wedge \mathbf{R}^4$, i.e., $\epsilon - \epsilon' = w\omega$. Using the commutativity again, one sees that if $w\omega' = \mathrm{d}\epsilon'$, then $w(\omega - \omega') = w(\mathrm{d}\Lambda)$, and so that $\omega - \omega' = \mathrm{d}\Lambda$. Thus the element R of $C_2(K) \otimes (\mathbf{R}^4 \wedge \mathbf{R}^4)$ is defined uniquely by $R = \mathrm{d}\omega$, independently of the choice of ϵ. This uses the fact that the horizontal rows of mappings are sequences, i.e., $d^2 = 0$. Using this fact again, we see that $\mathrm{d}R = 0$, and that $wR = w\mathrm{d}\omega = \mathrm{d}w\omega = \mathrm{d}^2\epsilon = 0$. □

R is the curvature of the metric h. We call the equation $\mathrm{d}R = 0$ the Bianchi identity, by analogy with §2. Likewise the equation $wR = 0$ is analogous to the equation $WR = 0$ of §2. It enforces the same symmetry property of the curvature tensor distribution. One can see from sequence (25) that R is a sum of elements of the form $\sigma \otimes t_{ab}$, where σ is a simplex, and the tensor t_{ab} lies in the one-dimensional subspace $\mathbf{R}^2_\perp \wedge \mathbf{R}^2_\perp$ of $\mathbf{R}^4 \wedge \mathbf{R}^4$. Thus t is a generator of rotations in the plane orthogonal to σ, leaving vectors tangent to σ fixed.

3.7 Calculating the metric from the curvature

Suppose that an element $R \in C_2(K) \otimes (\mathbf{R}^4 \wedge \mathbf{R}^4)$ such that $\mathrm{d}R = wR = 0$ is given. In order to determine a metric h which has this as its curvature, some extra information is needed to that assumed so far: one has to know that the horizontal sequences are exact.

Proposition If $H^2(K, L) = 0$, then $\mathrm{Im}\,(\varphi_2) =$ *curvatures*. If also $H^1(K, L) = H^0(K, L) = 0$, then $\ker(\varphi_2) \simeq C^0(K, L) \otimes \mathbf{R}^4$

Proof: In §5, we show that the constraints on the topology imply the exactness of the horizontal sequences. Here we assume that these conditions are met. Thus we can assume the existence of an $\omega \in C_3(K) \otimes (\mathbf{R}^4 \wedge \mathbf{R}^4)$ such that $\mathrm{d}\omega = R$, and that if ω' also satisfies this condition, then $\omega - \omega' = \mathrm{d}\Lambda$, for $\Lambda \in C_4(K) \otimes \mathbf{R}^4 \wedge \mathbf{R}^4$. Now $\mathrm{d}w\omega = wR = 0$, and so again using horizontal exactness, there exists $\epsilon \in C_{1,4}(K) \otimes \mathbf{R}^4$ such that $\mathrm{d}\epsilon = w\omega$. Moreover, if also $\mathrm{d}\epsilon' = w\omega' = w(\omega - \mathrm{d}\Lambda) = \mathrm{d}\epsilon - \mathrm{d}w\Lambda$, then $\epsilon - \epsilon' - w\Lambda = \mathrm{d}\alpha$ for an element $\alpha \in C^0(K, L) \otimes \mathbf{R}^4$. An element $h \in B_4(K)$ is defined by $h = s\epsilon$. For a different choice of ω and ϵ, ω' and ϵ', one finds $h' = s\epsilon' = h - s\mathrm{d}\alpha$, since $sw\Lambda = 0$. Thus the curvature R defines an element h of $B_4(K)$, arbitrary up to the addition of $s\mathrm{d}\alpha$, for any $\alpha \in C^0(K, L) \otimes \mathbf{R}^4$. Moreover $\partial' h = \partial' s\epsilon = -s\mathrm{d}\epsilon = -sw\omega = 0$, and so h defines a linearised metric. □

The gauge freedom $C^0(K, L) \otimes \mathbf{R}^4$ represents the linearised 'diffeomorphisms', or motions of the interior vertices of K in flat space.

3.8 The ‘fundamental theorem’

The theory presented so far, with the exception of the proof of the exactness of the horizontal sequences, almost completes a proof of the ‘fundamental theorem’ (Barrett 1987b).

Theorem Let (K,L) be a simplicial 4-manifold with boundary such that $H^n(K,L) = 0$ for $n = 0, 1, 2$. If *metrics* denotes the kernel of ∂' in $B_4(K)$, *translations* the space $C^0(K,L) \otimes \mathbf{R}^4$, *curvatures* the intersection of the kernels of w and d in $C_2(K) \otimes (\mathbf{R}^4 \wedge \mathbf{R}^4)$, then

$$0 \longrightarrow \mathit{translations} \xrightarrow{\varphi_1} \mathit{metrics} \xrightarrow{\varphi_2} \mathit{curvatures} \longrightarrow 0 \qquad (27)$$

is exact. The maps are $\varphi_1 = sd$ and φ_2 the mapping described above in §3.6, namely $\varphi_2 = \mathrm{d}(w^{-1})\mathrm{d}(s^{-1})$.

Proof: §3.7 shows that φ_2 is onto and that the kernel is the image of φ_1. (The ‘different choices’ of ω and ϵ in §3.7 are the only possible ones which have the same curvature.) It remains to show that φ_1 is injective. Suppose $\alpha \in C^0(K,L) \otimes \mathbf{R}^4$. Then if $s\mathrm{d}\alpha = 0$, then there exists $\Lambda \in C_4(K) \otimes (\mathbf{R}^4 \wedge \mathbf{R}^4)$ such that $w\Lambda = \mathrm{d}\alpha$. Now $w\mathrm{d}\Lambda = \mathrm{d}w\Lambda = \mathrm{d}^2\alpha = 0$, and so $d\Lambda = 0$, since the w involved here is injective.

The horizontal exactness of the sequences in (22), which we are assuming for the purposes of this proof, implies that $\Lambda = 0$ and so $\mathrm{d}\alpha = 0$. Again the horizontal exactness gives $\alpha = 0$. Hence φ_1 is injective. □

The horizontal exactness is proved in §5. The way in which the boundary comes in is important for this result. If, for example, the theory had been formulated on an infinite simplicial complex covering the whole of M, then constant translations and rotations would not alter the metric or connection, and would spoil the exactness of the horizontal sequences. One expects the result to be true in this case if one factors these linearised isometries out of the space $C^0(K,L) \otimes \mathbf{R}^4$, or, equivalently, includes another term on the left hand end of the sequences to take these into account.

3.9 The deficit angles

The bottom right hand corner of (22) can be extended so that a space which is a direct representation of the linearised deficit angles can be introduced. The space $D = \bigoplus_\sigma \left(C_2(K(\sigma)) \otimes (\mathbf{R}^2_\perp \wedge \mathbf{R}^2_\perp)\right)$, where the summation is over 2-simplexes, is the subspace of $C_2(K) \otimes (\mathbf{R}^4 \wedge \mathbf{R}^4)$ generated by elements $\sigma \otimes t_{ab}$ such that t_{ab} is orthogonal to the 2-simplex σ. The space $\bigoplus_\sigma \left(C_1(K(\sigma)) \otimes (\mathbf{R}^3_\perp \wedge \mathbf{R}^3_\perp)\right)$, where one sums over 1-simplexes, has a similar interpretation.

$$\begin{array}{ccccc}
 & \cdots & & & \\
 & \uparrow w & & & \\
\cdots \xrightarrow{d} & C_2(K) \otimes \mathbf{R}^4 \wedge \mathbf{R}^4 & \xrightarrow{d} & C_1(K) \otimes \mathbf{R}^4 \wedge \mathbf{R}^4 & \\
 & \uparrow & & \uparrow & \\
 & \bigoplus_\sigma \left(C_2(K(\sigma)) \otimes (\mathbf{R}^2_\perp \wedge \mathbf{R}^2_\perp)\right) & \xrightarrow{d} & \bigoplus_\sigma \left(C_1(K(\sigma)) \otimes (\mathbf{R}^3_\perp \wedge \mathbf{R}^3_\perp)\right) & \\
 & \uparrow & & \uparrow & \\
 & 0 & & 0 &
\end{array}$$

The two vertical sequences are multiple copies of (parts of) (25) and (26). Since the curvature $R \in C_2(K) \otimes (\mathbf{R}^4 \wedge \mathbf{R}^4)$ obeys $wR = \mathrm{d}R = 0$, it is equal to a unique element $\Delta \in D$ satisfying $\mathrm{d}\Delta = 0$. Since $\mathbf{R}^2_\perp \wedge \mathbf{R}^2_\perp \simeq \mathbf{R}$, the information contained in Δ is one real number per 2-simplex. We shall show in §4 that these are precisely the linearised deficit angles. The particular identification of a coefficient of σ in $\Delta = \sum \sigma \otimes t_{ab}$ with a linearised deficit angle for σ is dealt with in §4.

One can see that the Bianchi identities, $\mathrm{d}\Delta = 0$, are equivalent to 3 equations per 1-simplex.

4 Linearising the non-linear theory

Returning to the definition of a configuration of linearised Regge calculus given at the beginning of §3, we can now see that both a general metric $\gamma \in G$ and a linearised metric h can be represented as elements of $B_4(K)$, namely the element $\sum \sigma \otimes b(\sigma)$, where one sums over the 4-simplexes σ in K (which are given the standard orientation mentioned above), and $b(\sigma)$ is the bilinear form of the metric, or linearised metric, for the simplex. Since the metrics agree on common 3-faces of two neighbouring 4-simplexes (and vanish when restricted to the boundary), one has $\partial'\gamma = 0$, and similarly $\partial' h = 0$. One should note that the metric γ, regarded as a element of $B_4(K)$, is not generally a continuous function. It has discontinuities on the 3-simplexes. If $\gamma(t) \in B_4(K)$ is a one-parameter family of metrics with $\gamma(0) = g$, then $h = \mathrm{d}\gamma/\mathrm{d}t(0)$ defines a linearised metric.

Clearly once a linearised metric is defined in this way, one can make all the calculations of §3, according to the diagram (22), and naming the quantities linearised tetrad, connection, curvature, etc.. However in this section we wish to raise the question of the relation of these quantities to limits of their non-linear counterparts. One particular issue is the identification of the linearised deficit angles: given the one parameter family of metrics $\gamma(t)$, $t \in \mathbf{R}$, one can calculate the deficit angles $\delta(t)$ for each 2-simplex. How does $\dot{\delta} = \mathrm{d}\delta/\mathrm{d}t(0)$ relate to the linearised curvature $R \in C_2(K) \otimes (\mathbf{R}^4 \wedge \mathbf{R}^4)$ of linearised metric h?

4.1 Non-linear geometry

One can see that, given a metric γ, one can introduce constant tetrads $\theta^a(\sigma)$ on each simplex, which reduce to elements of $C_{1,4}(K) \otimes \mathbf{R}^4$ in the linearised limit, related by the mapping s to the linearised metric. The (torsion-free) connection for γ can be represented by linear mappings in $SO(4)$ at the 3-faces between two 4-simplexes, and on the boundary 3-simplexes. The definitions of the connection and curvature, and other aspects of the geometry, can be found in a previous paper (Barrett 1987a). The linear mappings in SO(4) are defined below (§4.2). The treatment of the boundary is special to this paper. Any configuration $\gamma \in G$ can be considered a metric for the whole of $\mathbf{R}^4$: γ defines the metric on M, and the flat metric g defines the metric on the complement. This defines a metric on $\mathbf{R}^4$ because the two metrics agree on the boundary ∂M. Thus a unique parallel transport for vectors across a boundary 3-simplex from M to the surrounding $\mathbf{R}^4 - M$ is defined.

Similarly, the curvature is represented by linear mappings in $SO(4)$ for each two-simplex (the holonomy elements expressed in any one particular frame). In the case of the boundary 2-simplexes one measures the holonomy for loops which are partly in $\mathbf{R}^4 - M$, crossing the boundary ∂M at the two boundary 3-simplexes bounding the 2-simplex. Thus the deficit angles for interior 2-simplexes are the normal deficit angles, but for boundary 2-simplexes they are the difference in the extrinsic curvatures of the inside and outside of ∂M (Hartle and Sorkin 1981). Alternatively, one can think of a boundary deficit angle as an ordinary (interior) deficit angle for the extended geometry discussed above, which is flat space outside of M. In fact all of the geometrical quantities associated with the boundary can be thought of in this way. One imagines that M is in the interior of a larger piecewise-flat manifold, where the outside is in fact flat, and moreover where the metric and tetrad outside do not vary with the parameter t of the one-parameter family. Then one just computes all the interior geometric quantities (connection, curvature, etc.) for this larger triangulation.

4.2 The equations

Inspection of sections 2 and 3 shows that the equations relating the tetrad, connection and curvature to the metric are, in the case of linearised Regge calculus, just the same equations which one has in linearised general relativity, extended from differential forms to currents of the same degree. One might assume, therefore, that in Regge calculus one can get these equations just by linearising the appropriate non-linear equations for Regge calculus. (The equations of the non-linear geometry are developed in a previous paper (Barrett 1987a).) This might actually be true, but it turns out to be somewhat difficult to implement, because although one can express the metric $\gamma(t)$ as a current of degree 0, as above, the equations one has are difficult to express as relations between currents, such that they reduce to those of (22) in the linearised limit. One particular problem is that it is hard to organise the signs of all the quantities without introducing a large amount of extra work and notation.

Let $\boldsymbol{\tau}$ be a particular 3-simplex, and $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ the two 4-simplexes for which $\boldsymbol{\tau}$ is a face, in a particular order $(\boldsymbol{\sigma}, \boldsymbol{\sigma}')$, this order corresponding to picking the 'direction of travel' across the face. If $\boldsymbol{\tau} \subset \partial M$, then one of $\boldsymbol{\sigma}$ or $\boldsymbol{\sigma}'$ denotes $\mathbf{R}^4 - M$. The connection at this face is defined by the parallel transport operator $\Omega_a{}^b \in SO(4)$, which is uniquely determined as a function of the tetrads $\theta^a(\boldsymbol{\sigma})$ and $\theta^a(\boldsymbol{\sigma}')$. We put $\theta^a(\mathbf{R}^4 - M) = e^a$. The definition of $\Omega_a{}^b$ is (Barrett 1987a)

$$i_* \left(\Omega_b{}^a \theta^b(\boldsymbol{\sigma}) - \theta^a(\boldsymbol{\sigma}')\right) = 0 \tag{28}$$

where i_* restricts the 1-form to the face. When one 'linearises' this relation, i.e. takes the derivative at $t = 0$ of an appropriate one-parameter family, one finds

$$i_* \left(\omega_b{}^a e^b + \epsilon^a(\boldsymbol{\sigma}) - \epsilon^a(\boldsymbol{\sigma}')\right) = 0 \tag{29}$$

where $\omega = \mathrm{d}\Omega/\mathrm{d}t(0), e = \theta(0), \epsilon = \mathrm{d}\theta/\mathrm{d}t$. If $\boldsymbol{\tau}$ is the face, τ a corresponding oriented simplex with one of the two different orientations, then (29) is equivalent to

$$\tau \wedge \left(\omega_b{}^a e^b + \epsilon^a(\boldsymbol{\sigma}) - \epsilon^a(\boldsymbol{\sigma}')\right) = 0 \tag{30}$$

This equation is consistent with one which one gets from the theory of §3,

$$w\left(\sum \tau \wedge \omega(\boldsymbol{\tau})\right) = \mathrm{d}\left(\sum \sigma \wedge \epsilon(\boldsymbol{\sigma})\right) \tag{31}$$

at least if one can show that the signs of the terms agree. The left-hand summation is over all geometric 3-simplexes, with τ a corresponding oriented simplex with some arbitrary choice of orientation, and the right- hand side over all geometric 4-simplexes, with the standard orientation chosen.

In a similar way one can sees that the other parts of (22) correspond to the 'linearisation' of their non-linear counterparts, up to signs. We shall show that the signs can be consistently identified so that the equations connecting the quantities in (22) are the linearisations of the non-linear equations. This done by making a *gauge transformation* on the manifold $\mathbf{R}^4 - M_2$, the manifold obtained by removing the 2-skeleton M_2, so that the connection is an ordinary function satisfying (1). A rotational gauge transformation can be employed to render the connection a piecewise-continuous function on $\mathbf{R}^4 - M_2$, or in conjuction with diffeomorphisms, a C^∞ function on $\mathbf{R}^4 - M_2$. The linearised limit of the tetrad, connection and other quantities on $\mathbf{R}^4 - M_2$, and the equations between them, is then identical to that of §2 for linearised general relativity. The *linearised* equations now have extensions to general currents, and one can use the inverse of the linearisation of the gauge transformations introduced to relate the linearised quantities obtained to the currents one calculates directly with (22).

In particular, the gauge transformations introduced do not affect gauge invariant quantities such as the deficit angles. If $R \in C_2(K) \otimes (\mathbf{R}^4 \wedge \mathbf{R}^4)$ is the linearised curvature, $\omega \in C_3(K) \otimes (\mathbf{R}^4 \wedge \mathbf{R}^4)$ the linearised connection defined directly by (22), $\bar{\omega}$ a linearised connection 1-form on $\mathbf{R}^4 - M_2$ defined by the above 'smoothing' procedure, and f a test function (a 2-form of compact support) then

$$R_a{}^b[f] = \omega_a{}^b[df] = \int_{\mathbf{R}^4 - M_2} \bar{\omega}_a{}^b \wedge \mathrm{d}f \tag{32}$$

since $\omega - \bar{\omega} = \mathrm{d}\Lambda$ on $\mathbf{R}^4 - M_2$. We suppose now, for simplicity, that f intersects just one 2-simplex, and arrange coordinates r, φ, x^2, x^3 such that r and φ are polar coordinates for the plane orthogonal to the 2-simplex, and x^2, x^3 are coordinates along it. Further, we suppose that (r, φ, x^2, x^3) is a positively-oriented coordinate system, and that $e^0 = \mathrm{d}r$, $e^1 = r\mathrm{d}\varphi$, $e^2 = \mathrm{d}x^2$, $e^3 = \mathrm{d}x^3$ form an orthonormal tetrad. One can easily verify that the linearised deficit angle $\dot{\delta}$ is given by

$$\dot{\delta} = -\int \bar{\omega}_0{}^1 \tag{33}$$

where the integral is in a direction of increasing φ ($\int \mathrm{d}\varphi > 0$) in a loop encircling the plane of the simplex once. The integral of (32) is equal to the limit $r \to 0$ of

$$-\int_{\partial(\mathbf{R}^4 - M_2)} \bar{\omega}_a{}^b \wedge f \tag{34}$$

where we suppose that $\mathbf{R}^4 - M_2$ has a boundary consisting of a small cylinder of radius r about $r = 0$. Integral (34) splits into a one-dimensional integral around the cylinder and a two-dimensional integral along it, giving

$$R_0{}^1[f] = \dot{\delta} \int_\sigma f \tag{35}$$

where, juggling the orientations, one finds that σ has an orientation agreeing with that of the coordinates $\sigma \to (x^3, x^2)$. Thus the *contribution* of this simplex to the linearised curvature is $R_0{}^1 = \dot{\delta}\sigma$, with this orientation for σ. Clearly $R_1{}^0 = -\dot{\delta}\sigma$, and the other components in this frame are zero.

5. The horizontal sequences

This section is devoted to calculating the homology of sequence (21), thus completing the proofs of §3.7, §3.8. To be specific, we show that the homology groups of (21) are the usual simplicial homology groups of M (in the order $H_n, \ldots H_0$), and that if M is topologically trivial, the horizontal sequences of (22) are exact.

In this section n and the dimension d of the manifold are allowed to be general, although only the results for $d = 4$ and $n = 1$ (and the trivial $n = 0$) are needed in the earlier sections.

Some standard notions of simplicial topology are required. (For more details see Maunder (1970).) The derived complex of a complex K is denoted K'. The simplexes of K' are $(\hat{\sigma}_1, \hat{\sigma}_2, \ldots, \hat{\sigma}_s)$, where $\hat{\sigma}_i$ is the centre of simplex $\boldsymbol{\sigma}_i$ of K, and $\boldsymbol{\sigma}_i$ is a proper face of $\boldsymbol{\sigma}_j$ for $i < j$. If $\boldsymbol{\sigma}$ is a simplex, $\dot{\sigma}$ denotes the complex formed by the proper faces of $\boldsymbol{\sigma}$, the boundary of $\boldsymbol{\sigma}$. The r-skeleton of complex K is denoted K^r. It consists of the simplexes of K of dimension r or less. If a total ordering $\ll$ of the simplexes of K is introduced, then the cap product is defined for $r \le s$ as a map

$$C_s \otimes C^r \to C_{s-r}(K)$$

where the image of $\sigma \otimes \alpha$, written $\sigma \cap \alpha$, is, if $\sigma = (v_1, v_2, \ldots, v_s)$ and the vertices are in the correct order $v_1 \ll v_2 \ll, \ldots, \ll v_s$,

$$\sigma \cap \alpha = (v_1, v_2, \ldots, v_{s-r})\alpha(v_{s-r}, \ldots, v_s). \tag{36}$$

The image of a general element $\sum \sigma \otimes \alpha$ is defined by linearity. The following equation holds

$$\partial(\sigma \cap \alpha) = (\partial\sigma) \cap \alpha + (-1)^{s-r} \sigma \cap (\delta\alpha). \tag{37}$$

An r-block e and its boundary $\dot{e} \subset e$ are defined to be subcomplexes of K such that e has dimension r and $H_k(e, \dot{e}) = 0$ unless $k = r$. The interior of e is defined to be the set of simplexes in $e - \dot{e}$. A block dissection of K is defined to be a set of blocks such that each simplex of K is in the interior of one block, and such that the boundary of an r-block is contained in the union of the m-blocks for $m \le r$.

The r-skeleton of the block dissection, M^r, is then defined to be the union of the m-blocks, for $m \le r$. The sequence

$$\ldots \to H_{r+1}(M^{r+1}, M^r) \xrightarrow{\partial_*} H_r(M^r, M^{r-1}) \xrightarrow{\partial_*} H_{r-1}(M^{r-1}, M^{r-2}) \xrightarrow{\partial_*} \ldots \tag{38}$$

has the same homology as that of $K = M^d = M^{d+1} = \ldots$. The space $H_r(M^r, M^{r-1})$ is equal to $\oplus H_r(e, \dot{e})$, where the direct sum is over r-blocks e.

All of the above is standard in a simplicial treatment of duality, except that our blocks are more general in that we do not require that $H_r(e, \dot{e}) = \mathbf{R}$ (or $\mathbf{Z}$).

We now return to the complex K being the specific one in the triangulation (K, L) of $(M, \partial M)$ of the earlier sections. With a particular fixed value of n, $0 \leq n \leq d$, we introduce a total ordering of the vertices of K' such that

$$\hat{\sigma}_n \ll \hat{\sigma}_{n+1} \ll \ldots \ll \hat{\sigma}_d \ll \hat{\sigma}_{n-1} \ll \hat{\sigma}_{n-2} \ll \ldots \ll \hat{\sigma}_0 \tag{39}$$

for any vertices $\hat{\sigma}$ where the subscript indicates the dimension of the simplex $\boldsymbol{\sigma}$. The case $n = d$ is the usual ordering one uses to prove Poincaré-Alexander duality (this will be referred to as 'ordinary duality').

For each k-simplex $\boldsymbol{\sigma}$ we define the block $(e(\boldsymbol{\sigma}), \dot{e}(\boldsymbol{\sigma}))$ by

$$\begin{aligned} \dot{e}(\boldsymbol{\sigma}) &= \{(v_1, v_2, \ldots v_r) \in K' : v_1 \ll v_2 \ll \ldots \ll v_r \ll \hat{\sigma}\} \\ e(\boldsymbol{\sigma}) &= (\hat{\sigma}) * \dot{e}(\boldsymbol{\sigma}) \end{aligned} , \tag{40}$$

the $*$ denoting the join operation. These have a differing character for $k < n$ and $k \geq n$. For $k \geq n$, $e(\boldsymbol{\sigma}) \subset K(\boldsymbol{\sigma})$ and has dimension $k - n$, while for $k < n$ these are the cells dual to the simplex which one has in ordinary duality, having dimension $d - k$. In the case $k < n$ one has

$$\begin{aligned} H_m(e(\boldsymbol{\sigma}), \dot{e}(\boldsymbol{\sigma})) &\simeq 0 \quad \text{if} \quad \boldsymbol{\sigma} \in L, \quad \text{or} \quad \boldsymbol{\sigma} \in K - L \quad \text{and} \quad m + k \neq d \\ &\simeq \mathbf{R} \quad \text{if} \quad \boldsymbol{\sigma} \in K - L \quad \text{and} \quad m = d - k. \end{aligned} \tag{41}$$

The situation for $k \geq n$ is a little more complicated and departs from the ordinary theory. The result is the following:

$$\begin{aligned} H_m(e(\boldsymbol{\sigma}), \dot{e}(\boldsymbol{\sigma})) &\simeq 0 \quad \text{if} \quad m \neq k - n, \\ &\simeq \lambda^n(\boldsymbol{\sigma}) \quad \text{if} \quad m = k - n. \end{aligned} \tag{42 - 43}$$

Proof: As discussed in §3.3, $\lambda^n(\boldsymbol{\sigma})$ can be identified with the kernel of d in $C^n(K(\boldsymbol{\sigma}))$. Consider the ordinary (Poincaré) duality on the boundary sigma, i.e. using the standard ordering

$$\hat{\sigma}_{d-1} < \hat{\sigma}_{d-2} < \ldots < \hat{\sigma}_n < \hat{\sigma}_{n-1} < \ldots < \hat{\sigma}_0. \tag{44}$$

to define the cap product $\dot{\cap}$ the dot to distinguish it from $\cap$) and the dual blocks. We call the r-skeleton of this block dissection N^r, for $r \leq k - 1$. Then $\dot{e}(\boldsymbol{\sigma}) = N^{k-n-1}$ (including the case $\dot{e}(\boldsymbol{\sigma}) = \emptyset$ when $k = n$). The two sequences

$$\begin{array}{ccccccccccccc} 0 \xrightarrow{\mathrm{d}} & C^0(K(\boldsymbol{\sigma})) & \xrightarrow{\mathrm{d}} \ldots \xrightarrow{\mathrm{d}} & C^n(K(\boldsymbol{\sigma})) & \xrightarrow{\mathrm{d}} \ldots \xrightarrow{\mathrm{d}} & C^{k-1}(K(\boldsymbol{\sigma})) & \xrightarrow{\mathrm{d}} 0 \\ & \downarrow D & & \downarrow D & & \downarrow D & \\ 0 \xrightarrow{\partial_\bullet} & H_{k-1}(N^{k-1}, N^{k-2}) & \xrightarrow{\partial_\bullet} \ldots \xrightarrow{\partial_\bullet} & H_{k-n-1}(N^{k-n-1}, N^{k-n-2}) & \xrightarrow{\partial_\bullet} \ldots \xrightarrow{\partial_\bullet} & H_0(N^0) & \xrightarrow{\partial_\bullet} 0 \end{array} \tag{45}$$

are isomorphic by duality, i.e., each D is an isomorphism, and the diagram is commutative up to signs. They have the homology of $\dot{\sigma}$. It is convenient to extend the sequences at either ends, so that the homology of each sequence is the reduced homology/cohomology of $\boldsymbol{\sigma}$. (Thus all the homology groups vanish.) The spaces C^{-1} and C_{-1} are copies of $\mathbf{R}$.

$$\begin{array}{ccccccccccccccc} 0 \xrightarrow{\bar{\mathrm{d}}} & C^{-1} & \xrightarrow{\bar{\mathrm{d}}} & C^0(K(\boldsymbol{\sigma})) & \xrightarrow{\bar{\mathrm{d}}} \ldots \xrightarrow{\bar{\mathrm{d}}} & C^n(K(\boldsymbol{\sigma})) & \xrightarrow{\bar{\mathrm{d}}} \ldots \xrightarrow{\bar{\mathrm{d}}} & C^{k-1}(K(\boldsymbol{\sigma})) & \xrightarrow{\bar{\mathrm{d}}} & C^k(K(\boldsymbol{\sigma})) & \xrightarrow{\bar{\mathrm{d}}} 0 \\ & \downarrow D & & \downarrow D & & \downarrow D & & \downarrow D & & \downarrow D & \\ 0 \xrightarrow{\bar{\partial}_\bullet} & H_k(\boldsymbol{\sigma}, \dot{\sigma}) & \xrightarrow{\bar{\partial}_\bullet} & H_{k-1}(\dot{\sigma}, N^{k-2}) & \xrightarrow{\bar{\partial}_\bullet} \ldots \xrightarrow{\bar{\partial}_\bullet} & H_{k-n-1}(N^{k-n-1}, N^{k-n-2}) & \xrightarrow{\bar{\partial}_\bullet} \ldots \xrightarrow{\bar{\partial}_\bullet} & H_0(N^0) & \xrightarrow{\bar{\partial}_\bullet} & C_{-1} & \xrightarrow{\bar{\partial}_\bullet} 0 \end{array} \tag{46}$$

The mappings $\tilde{d}$ and $\tilde{\partial}$ are the same as d and ∂, except that the $\tilde{d}$ which acts on C^{-1} and the $\tilde{\partial}$ which maps C_0 into C_{-1} are the normal ones for a reduced sequence. The two mappings D on either end of the sequences can be defined so that the sequences are again isomorphic. For clarity of notation, we put $N^k = N^{k+1} = \ldots = K(\boldsymbol{\sigma})$, (and of course $N^{k-1} = \dot{\sigma}$), so that this is a block dissection of $\boldsymbol{\sigma}$. Let z be a generator of $C_k(K(\boldsymbol{\sigma}))$, and $\varphi: C_k(K(\boldsymbol{\sigma})) \to C_k(K(\boldsymbol{\sigma})')$ the subdivision chain map, so that $z' = \varphi(z)$ is a generator of $H_k(\boldsymbol{\sigma}, \dot{\sigma})$. The definition of the map D is then $\alpha \to \partial z' \dot{\cap} \alpha$ for $C^0 - \cdots C^{k-1}$, $\alpha \to \alpha z'$ on C^{-1}, and $\alpha \to \alpha(z')$ on C^k.

Now $(\ker \mathrm{d}) \subset C^n$ is mapped by D to $(\ker \tilde{\partial}_*) \subset H_{k-n-1}(N^{k-n-1}, N^{k-n-2})$ for $0 \le n \le k-1$, (and to C_{-1} for $n = k$) which is the $(n-k-1)$-th homology group of the sequence

$$0 \xrightarrow{!} H_{k-n-1}(N^{k-n-1}, N^{k-n-2}) \xrightarrow{\tilde{\partial}_*} \ldots \xrightarrow{\tilde{\partial}_*} H_1(N^1, N^0) \xrightarrow{\tilde{\partial}_*} H_0(N^0) \xrightarrow{\tilde{\partial}_*} C^{-1} \to 0 \tag{47}$$

Now the homology of (47) is the reduced homology of $\dot{e}(\boldsymbol{\sigma})$, since it is the (reduced) chain complex of a block dissection of it. All the other homology groups of (47) vanish because they are identical to those of (46). But because $e(\boldsymbol{\sigma})$ is topologically trivial, $H_m(e(\boldsymbol{\sigma}), \dot{e}(\boldsymbol{\sigma})) \simeq \tilde{H}_{m-1}(\dot{e}(\boldsymbol{\sigma}))$, and so the proof of (42) and (43) is complete.

We now turn to describing the isomorphism of (43) in terms of the duality induced by the cap product of (36). For this, we return to using the ordering of (39) in the cap product. We define the map

$$\Gamma: C^m(K(\boldsymbol{\sigma})', K(\boldsymbol{\sigma})'^{n-1}) \to C_{k-m}(K(\boldsymbol{\sigma})'), \qquad 0 \le m \le k$$

$$\alpha \mapsto \varphi(z) \cap j(\alpha)$$

where $z \in C_k(K(\boldsymbol{\sigma}))$ is fixed as before and j is the standard injection $C^m(K(\boldsymbol{\sigma})', K(\boldsymbol{\sigma})'^{n-1}) \to C^m(K(\boldsymbol{\sigma})')$. One can show fairly easily that Γ induces a map Γ_*

$$\Gamma_*: H^n(\boldsymbol{\sigma}, K(\boldsymbol{\sigma})^{n-1})) \to H_{k-n}(e(\boldsymbol{\sigma}), \dot{e}(\boldsymbol{\sigma}))$$

Theorem: Γ_* is a linear isomorphism.

Proof: Consider the composite mapping γ:

$$\tilde{H}^{n-1}(K(\boldsymbol{\sigma})^{n-1}) \xrightarrow{\tilde{\partial}_*} H^n(\boldsymbol{\sigma}, K(\boldsymbol{\sigma})^{n-1})) \xrightarrow{\Gamma_*} H_{k-n}(e(\boldsymbol{\sigma}), \dot{e}(\boldsymbol{\sigma})) \xrightarrow{\tilde{\partial}_*} \tilde{H}_{k-n-1}(\dot{e}(\boldsymbol{\sigma}))$$

i.e., $\gamma = \tilde{\partial}_* \Gamma_* \tilde{\partial}^*$. Using (37), one can show that this mapping is

$$\begin{aligned} [\beta] &\mapsto (-1)^{k-n} \tilde{\partial}(\partial z' \cap \beta), \qquad && k \ge n > 0 \\ \beta &\mapsto \beta \partial z', && n = 0 \end{aligned} \tag{48}$$

where $[\beta]$ denotes the equivalence class of $\beta \in C^{n-1}(K(\boldsymbol{\sigma})'^{n-1})$ in $\tilde{H}^{n-1}(K(\boldsymbol{\sigma})^{n-1})$. Despite the different ordering used in the duality of (46), this map is the same, up to the sign, as the mapping induced by $\partial_* D$ in (46), i.e., $[\beta] \mapsto \pm \tilde{\partial}_* D \beta$, for all n, $0 \le n \le k$. Since

$$\tilde{H}^{n-1}(K(\boldsymbol{\sigma})^{n-1}) = C^{n-1}/\mathrm{Im}\,\tilde{d} \;\overset{D}{\simeq}\; H_{k-n}(N^{k-n}, N^{k-n-1})/\mathrm{Im}\,\tilde{\partial}_* \;\overset{\tilde{\partial}_*}{\simeq}\; \ker \tilde{\partial}_*,$$

one concludes that γ is an isomorphism, and hence that Γ_* is also.

There is a natural isomorphism between $H^n(\boldsymbol{\sigma}, K(\boldsymbol{\sigma})^{n-1}))$ and $\lambda^n(\boldsymbol{\sigma})$, because the former is the n-th homology group of the sequence

$$0 \xrightarrow{!} C^n(K(\boldsymbol{\sigma})) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} C^{k-1}(K(\boldsymbol{\sigma})) \xrightarrow{\mathrm{d}} C^k(K(\boldsymbol{\sigma})) \xrightarrow{\mathrm{d}} 0 \tag{49}$$

and hence equal to $(\ker \mathrm{d}) \subset C^n(K(\boldsymbol{\sigma}))$.

5.1 The homology of sequence (21)

The map Γ_* depended on an arbitrary choice of a generator z of $C^k(K(\boldsymbol{\sigma}))$. If we include this explicitly as an argument, we have a linear isomorphism

$$\Gamma' : C^k(K(\boldsymbol{\sigma})) \otimes H^n(\boldsymbol{\sigma}, K(\boldsymbol{\sigma})^{n-1})) \to H_{k-n}(e(\sigma), \dot{e}(\sigma))$$

Thus, if $d \geq k \geq n$, taking the direct sum over k-simplexes, Γ' induces a map

$$\Gamma' \colon C_{n,k}(K) \quad \to \quad H_{k-n}(M^{k-n}, M^{k-n-1}).$$

Also, for $0 \leq k < n$ one can define an isomorphism $\Gamma' \colon C^k(K, L) \to H_{d-k}(M^{d-k}, M^{d-k-1})$, in the same way as for ordinary duality (Again the reader is referred to Maunder (1970)). In fact one can use the ordering (39), this giving an identical result. It is a fairly straightforward exercise to show that $\partial_* \Gamma' = \pm \Gamma' \mathrm{d}$ in all cases $0 \leq k \leq n$, so that the sequence (21) is isomorphic to (38) up to signs. Thus the homology of (21) is $H_*(K)$.

The case $n = 0$ is a proof of Lefschetz duality: $H^*(K, L) \simeq H_*(K)$.

5.2 The fundamental theorem

With these results, we can now see that in §3.7, §3.8 one needs to have $H^0(K, L) = H^1(K, L) = H^2(K, L) = 0$ for the conclusions to hold. This is certainly true in four dimensions if M is a topologically trivial region. For any dimension $d \geq 3$ this conclusion is also true, although one has to modify the diagrams in §3 slightly. For $d = 2$, $H^2(K, L) = H_0(K) = \mathbf{R}$ and so one can only compute the metric from a curvature if three extra real conditions hold, that the curvature R defines the zero element of $H_0(K) \otimes (\mathbf{R}^2 \wedge \mathbf{R}^2)$, and that $w\omega$ defines the trivial element of $H_0(K) \otimes \mathbf{R}^2$. These conditions were mentioned in a previous letter (Barrett 1987b)

[1] Not at this point necessarily satisfying any field equations.

[2] More generally, one sees that there are ten quantities which must vanish for each generator of $H^2(U, \mathbf{R})$ in order that ${R_a}^b$ should come from a metric. These are Penrose and Rindler's (1986 §6.4) 'ten vanishing integrals'

[3] A sequence of spaces and maps is exact if for every space not on an end of the sequence, the kernel of the map out of it is equal to the image of the map into it. There are no conditions associated with a space that ends a sequence.

[4] A *geometric simplex* is a certain convex subset of $\mathbf{R}^4$, whereas an *oriented simplex* is a map from a standard copy of a (geometric) simplex to the manifold in question (see Maunder (1970) for an exact definition of a simplicial chain group). Often the term *simplex* is used on its own, assuming it is clear which of these two is meant. In formulae, a boldface $\boldsymbol{\sigma}$ denotes a geometric simplex, ordinary typeface (σ) an oriented simplex.

[5] There is a more sophisticated definition of tensor distributions where the test function space is the space of sections of compact support of the appropriate vector bundle (Choquet-Bruhat, Dewitt-Morette, Dillard-Bleick 1977). We do not need this generality, as our manifold is a subset of $\mathbf{R}^4$.

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