# Dichromatic state sum models for four-manifolds from pivotal functors

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**Abstract**

A family of invariants of smooth, oriented four-dimensional manifolds is defined via handle decompositions and Kirby calculus. It is shown that some of them are stronger than the signature and Euler invariant. The invariants are parameterised by a pivotal functor from a spherical fusion category into a ribbon fusion category. A state sum formula for the invariant is constructed via a chain mail procedure, so a large class of topological state sum models can be phrased in terms of it. Most prominently, the Crane-Yetter state sum over an arbitrary ribbon fusion category is recovered, including the nonmodular case. Another example is the four-dimensional Dijkgraaf-Witten model. Derivations of state space dimensions of some TQFTs as special cases agree with recent calculations of ground state degeneracies in Walker-Wang models. It is also shown that the Crane-Yetter invariant for nonmodular categories is stronger than signature and Euler invariant. Relations to different approaches to quantum gravity such as Cartan geometry and teleparallel gravity are also discussed.

## 1 Introduction

The Crane-Yetter model [CKY97] is a state sum invariant of four-dimensional manifolds that determines a topological quantum field theory (TQFT). The purpose of this paper is to give a more general construction that puts the Crane-Yetter model in a wider context and allows the exploration of new models, as well as a more thorough understanding of the Crane-Yetter model itself. There is interest in four-dimensional TQFTs from solid-state physics, where they allow the study of topological insulators, for example in the framework of Walker and Wang [WW12], which is expected to be the Hamiltonian formulation of the Crane-Yetter TQFT. The Crane-Yetter model is also the starting point for constructing spin foam models of quantum gravity [BC98]. Therefore the main motivation for this paper is to provide a firmer and more unified basis for a variety of physical models.

A state sum model is a path integral formulation for a lattice theory. In order to calculate the transition amplitude from one lattice state to another (possibly on a different lattice), a cobordism, or spacetime, from the initial to the final lattice is discretised using a triangulation or a cell complex. Then the amplitude is the sum of a weight function over states on the discretised cobordism. A state is typically a labelling of the elements of the discretisation with some algebraic data, for example objects and morphisms in a certain category. In a topological state sum model, the sum over all states is independent of the particular discretisation chosen, and thus gives rise to a TQFT. The weight function corresponds to an action functional and is calculated locally, for example per simplex if the discretisation is a triangulation. This property is motivated by the physical assumption of the action being local. This is expected to have the far-reaching mathematical consequence that the resulting TQFT is 'fully extendible', which means that it is well-defined on manifolds with corners of all dimensions down to zero.

Topological state sum models are an approach for quantum gravity. The Turaev-Viro state sum is an excellent model of 3d Euclidean quantum gravity, and as Witten famously remarks [Wit89, section 3], one would expect any diffeomorphism-covariant theory to have a topological quantum theory. So far, no topological state sum has modelled 4d quantum gravity in a completely satisfactory way. The most prominent topological state sum model remains the $U_q sl(2)$-Crane-Yetter state sum; however this is not considered a gravity model. It was shown to reduce to the signature [CKY97] and the Reshetikhin-Turaev theory on the boundary [BFG07]. As a consequence of this, the dimensions of the state spaces attached to the boundary manifolds are only one-dimensional, whereas in a gravity theory one would expect a large state space containing many graviton modes. The more general framework developed here suggests some different Crane-Yetter type models that may be related to approaches such as teleparallel gravity [BW12].

## 1.1 The Crane-Yetter invariant and its dichromatic generalisation

In three-dimensional topology, the Turaev-Viro state sum invariant distinguishes even some homotopy-equivalent three-manifolds. However the Crane-Yetter invariant of four-manfolds for modular categories, as it was originally defined, is just a function of the signature and the Euler characteristic of the manifold.

A closer look at the construction reveals a possible explanation why this is the case. By the Morse theorem, smooth manifolds admit handle decompositions. Different handle decompositions of the same manifold can be related by a sequence of handle slides and cancellations. One way to construct a manifold invariant is thus assigning numbers to handle decompositions that are invariant under the handle moves. For example, there is a canonical handle decomposition determined by a triangulation, by thickening the dual complex. Handle decompositions can be described by Kirby diagrams. These are framed links where the components of the link represent the handles. For the modular Crane-Yetter invariant, the components of the link are each labelled by the Kirby colour of the ribbon fusion category $\mathcal{C}$ that determines the invariant. By the universal property of the tangle category [Shu94], this can be interpreted as diagrammatic calculus in that category. Evaluating the diagram and multiplying by a normalisation gives the invariant.

Since 2-handles are treated in the same way as 1-handles, there is a redundancy in the construction of the modular Crane-Yetter invariant $CY$: the modular Crane-Yetter invariant does not change if 1-handles are replaced by 2-handles with the same link diagram. This changes the topology of the manifold and ensures, for example, that every manifold has the same modular Crane-Yetter invariant as a simply-connected one. The invariant cannot even detect the first homology.

The general problem is to define invariants that label the 1- and 2-handles with different objects of the category. Petit's "dichromatic invariant" [Pet08], does exactly this: in addition to choosing the ribbon fusion category, one also chooses a ribbon fusion subcategory and labels the 2-handles with the Kirby colour of the subcategory. It will be shown in section 6.2 that this change does indeed lead to a stronger invariant that can distinguish manifolds with the same signature and Euler characteristic. As a bonus, the general Crane-Yetter invariant (for possibly nonmodular ribbon categories) is recovered as a special case of the dichromatic invariant. Previously, no description of it in terms of Kirby calculus was known. Now one can indeed pinpoint the improvement of the invariant as due to the differing labels on 1-handles and 2-handles.

A generalisation of the dichromatic invariant is presented here and translated into a state sum model. Instead of a ribbon fusion subcategory, the generalisation is to use a pivotal functor from a spherical fusion to a ribbon fusion category. The 1-handles are still labelled with the Kirby colour of the target category but the 2-handles are labelled with the Kirby colour of the source category, with the functor applied to it.

### 1.2 Outline

In section 2, the common definitions such as spherical and ribbon fusion categories and their graphical calculus are recalled. Various notational conventions are established.

In section 3, the sliding lemma from spherical and ribbon fusion categories is generalised. The original lemma allows one to slide the identity morphism of any object over any other object that is encircled by the Kirby colour of the category. The generalised lemma applies to sliding over an object that is encircled by the image of a Kirby colour via a pivotal functor. This generalisation will be a key step in the proof of invariance 3.3 of the *generalised dichromatic invariant* (3.2.1) of smooth, oriented 4-manifolds. The section concludes with some general properties of the invariant and a motivating special case, Petit's dichromatic invariant (example 3.13).

Many functors lead to the same invariant, and a general situation in which this is the case is presented in section 4. This often leads to a simplification of the invariant, especially when the functor and both categories are unitary, or when the target category is modularisable.

If the target category of the functor is modularisable, which is often the case, the generalised invariant can also be cast in the form of a state sum. In section 5, this state sum formula (5.2.5) is derived using the chain mail technique.

Section 6 is a non-exhaustive survey of several different examples of the generalised dichromatic invariant. The Crane-Yetter state sum is recovered as a special case, both for modular and nonmodular ribbon fusion categories. For the nonmodular Crane-Yetter invariant, a chain mail construction was not previously known. A further special case is Dijkgraaf-Witten theory without a cocycle. It is shown that the invariant can be sensitive to the fundamental group. The Dijkgraaf-Witten example is generalised to group homomorphisms.

There is a discussion in section 7 of how the present framework could connect to Walker-Wang models and state sum models used in the study of quantum gravity such as spin foam models. Relations to Cartan geometry and teleparallelism are discussed as well.

Finally, a handy overview of the different known special cases of our generalised dichromatic invariant is given as a table in section 8, together with some comments on the results.

## 2 Preliminaries

### 2.1 Introduction to monoidal categories with additional structure

In mathematical physics, one encounters a multitude of linear monoidal categories with additional structure and functors preserving this structure. Usually, the category Vect of finite dimensional vector spaces over $\mathbb{C}$ serves as a trivial example for these. The additional structure often arises as special cases of higher categorical structures, for example, monoidal categories are bicategories with one object and braided categories are a special case of tricategories with one 1-morphism. This beautiful motivation is explained more closely in the literature, e.g. [SP11, section B.3]. Here the definitions are given in a closely related manner by discussing their suitability for graphical calculus. Monoidal categories are needed for a graphical calculus of one-dimensional tangles in two dimensions; similarly one needs the braided structure for evaluating tangle diagrams in three dimensions.

### 2.1.1 Semisimple and linear categories

**Definition 2.1.** A **$\mathbb{C}$-linear category** is a category enriched in $\mathrm{Vect}_{\mathbb{C}}$. If not mentioned otherwise, all categories in this article are $\mathbb{C}$-linear categories and all functors are **linear functors**, that is, functors in the enriched category. This implies they are linear on the morphism spaces and preserve direct sums.

**Definition 2.2.** An object $X \in \operatorname{ob}\mathcal{C}$ is called **simple** if $\mathcal{C}(X,X) \cong \mathbb{C}$.

*Example* 2.3.

- In Vect, $\mathbb{C}$ is the only simple object up to isomorphism.
- In $\mathrm{Rep}(G)$, the representation category of a group $G$, the simple objects are the irreducible representations.

Note that simple objects are called scalar objects in [Pet08].

**Definition 2.4.** A linear category $\mathcal{C}$ is called **semisimple** if it has biproducts, idempotents split (i.e. it has subobjects) and there is a set of inequivalent simple objects $\Lambda_{\mathcal{C}}$ such that for each pair of objects $X$, $Y$, the map

$$\Phi\colon \bigoplus_{Z\in\Lambda_{\mathcal{C}}} \mathcal{C}(X,Z)\otimes\mathcal{C}(Z,Y) \to \mathcal{C}(X,Y)$$

obtained by composition and addition is an isomorphism. If the set $\Lambda_{\mathcal{C}}$ is finite, then the category is called **finitely semisimple**.

For a simple object $Z$ and any object $X$, there is a bilinear pairing

$$(-,-)\colon \mathcal{C}(Z,X)\times\mathcal{C}(X,Z) \to \mathbb{C}$$

defined by

$$(f,g)\cdot 1_Z = gf$$

**Lemma 2.5.** In a semisimple category, the bilinear pairing is non-degenerate.

*Proof.* Write $\Phi^{-1}(1_X) = \oplus_Z \pi_Z$ with $\pi_Z = \sum_i \alpha_{Z,i}\otimes\alpha_Z^i \in \mathcal{C}(X,Z)\otimes\mathcal{C}(Z,X)$. Then a calculation shows that for every $f\colon X\to Z$ and every simple object $Z$, $\pi_Z$ satisfies the two snake identities

$$\sum_i \left(\alpha_Z^i, f\right)\alpha_{Z,i} = f = \sum_i \left(f, \alpha_Z^i\right)\alpha_{Z,i}$$

and so the bilinear form is non-degenerate. □

The requirements of biproducts and subobjects in definition 2.4 are not very restrictive. According to the discussion in [Mü03a], any category that satisfies all of the conditions of the definition of a semisimple category except for the existence of biproducts and subobjects can be embedded as a full subcategory of a semisimple category.

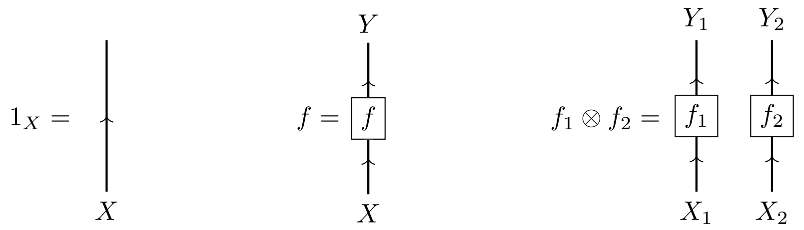
*Example* 2.6. For every finite group $G$, $\mathrm{Rep}(G)$ is finitely semisimple.

### 2.1.2 Monoidal categories with additional structure

An overview of most commonly used definitions of monoidal categories with additional structure, together with their graphical calculus can be found in [Sel09].

A **monoidal category** is a category $\mathcal{C}$ (linear by default) together with a bilinear functor $-\otimes-\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ (the $-$ are placeholders) called the **monoidal product**, and a unit object $I$ called the **monoidal identity**. There are also natural associativity and unit isomorphisms subject to coherence conditions which can be found e.g. in [Sel09]. In a **strict monoidal category**, the coherence morphisms are all identity morphisms.

In the graphical calculus, morphisms $f\colon X\to Y$ in a monoidal category are drawn as boxes and lines in the plane, from the bottom to the top



(2.1.1)

The upward-pointing arrow on the lines is optional at this point but will be a useful device when duals are introduced. The coherence morphisms are not shown in the diagrammatic calculus. This is due to MacLane's famous coherence theorem which states that any product of coherence morphisms between two objects is a unique morphism [ML63]. Hence there is no ambiguity in the way the coherence morphisms are inserted. Also, the coherence theorem shows that every monoidal category is monoidally equivalent to a strict monoidal category. Hence one can alternatively view the diagrammatic calculus as determining morphisms in the equivalent strict category. Throughout the paper, monoidal categories (possibly with extra structure) will be indicated by the name of the mere category whenever standard notation for all the additional data is used.

Monoidal functors also have coherence morphisms. In this paper, a monoidal functor means a functor in which these morphisms are isomorphisms (also called sometimes a strong monoidal functor).

**Definition 2.7.** A **monoidal functor** is a tuple $(F, F^2, F^0)$, where

- $F\colon \mathcal{C}\to\mathcal{D}$ is a functor,
- $F^2\colon F(-)\otimes_{\mathcal{D}} F(-)\Rightarrow F(-\otimes_{\mathcal{C}} -)$ is a natural isomorphism,
- $F^0\colon I_D\to F(I_C)$ is an isomorphism in $\mathcal{D}$.

$F^2$ and $F^0$ are required to commute with the coherence morphisms. A **monoidal natural transformation** is a natural transformation that commutes with $F^0$ and $F^2$.

### 2.1.3 Rigid and fusion categories

**Definition 2.8.** A **duality** is a tuple $(X, Y, \mathrm{ev}, \mathrm{coev})$ where $\mathrm{ev}\colon X\otimes Y\to I$, $\mathrm{coev}\colon I\to Y\otimes X$, satisfying the "snake identities":

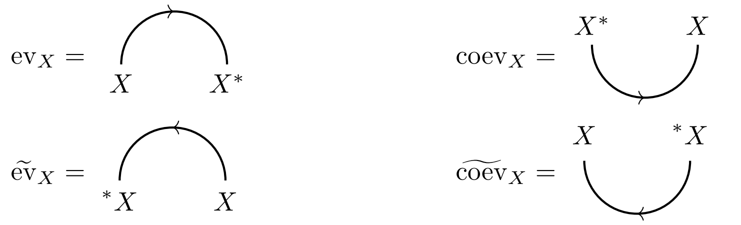
$$(\mathrm{ev}\otimes 1_X)\circ(1_X\otimes\mathrm{coev}) = 1_X$$
$$(1_Y\otimes\mathrm{ev})\circ(\mathrm{coev}\otimes 1_Y) = 1_Y \qquad (2.1.2)$$

In this situation, $(X, \mathrm{ev}, \mathrm{coev})$ is called the left dual of $Y$, and $(Y, \mathrm{ev}, \mathrm{coev})$ the right dual of $X$. The morphisms ev and coev are called evaluation and coevaluation, respectively. (When talking about adjunctions, these are also called "unit" and "counit".)

**Definition 2.9.** A monoidal category with left (right) duals for every object is called a **left (right) rigid category**. A **rigid**, or **autonomous** category is a category that is left rigid and right rigid, i.e., every object has a left and a right dual.

In the following, each object $X$ will have a particular choice of duals. The right dual is denoted $(X^*, \mathrm{ev}_X, \mathrm{coev}_X)$ and the left dual $({}^*X, \widetilde{\mathrm{ev}}_X, \widetilde{\mathrm{coev}}_X)$. This defines right and left dual monoidal functors $-^*$ and ${}^*-$ that are contravariant in the sense that source and target are switched and that the monoidal product is reversed.

Evaluation and coevaluation morphisms are drawn as caps and cups. The arrow in the diagram is an orientation for the line that points to the right for the right duals.




(2.1.3)

The arrow notation means that it is possible to regard the object $X$ as a label on the whole line (rather than one end of it). The convention at the ends of the line is that an upward-pointing arrow indicates $X$ and a downward-pointing arrow $X^*$.

In this graphical calculus, the snake identities now become:

(2.1.4)

Indeed, every identity of strings that is true as an isotopy in the plane is true for morphisms in a rigid category.

Applying a monoidal functor $\left(F, F^2, F^0\right)$ to the snake identities shows that dualities are preserved, i.e. that the following morphism is an evaluation:

$$FX \otimes FY \xrightarrow{F^2_{X,Y}} F\left(X \otimes Y\right) \xrightarrow{F\,\mathrm{ev}} FI_{\mathcal{C}} \xrightarrow{\left(F^0\right)^{-1}} I_{\mathcal{D}}$$

A similar statement holds for the coevaluation. Proving this requires all the naturality axioms of a monoidal functor.

A standard result on dualities is that any two duals of a given object $X$ are canonically isomorphic. Applying this to $F$ shows [Pfe09] that there are canonical isomorphisms for the right duals

$$u_X : F\left(X^*\right) \to \left(FX\right)^* \tag{2.1.5}$$

determined by $F$. These satisfy the defining equations

$$\mathrm{ev}_{FX} = \left(F^0\right)^{-1} \circ F\,\mathrm{ev}_X \circ F^2_{X,X^*} \circ \left(1 \otimes u_X^{-1}\right) \tag{2.1.6}$$

$$\mathrm{coev}_{FX} = \left(u_X \otimes 1\right) \circ \left(F^2_{X^*,X}\right)^{-1} \circ F\,\mathrm{coev}_X \circ F^0 \tag{2.1.7}$$

There are also separate canonical isormophisms in a similar way for the left duals.

**Definition 2.10.** Finitely semisimple rigid categories with simple $I$ are known as **fusion categories**.

### 2.1.4 Spherical pivotal categories

Since in every rigid category, there are canonical isomorphisms ${}^*(X^*) \cong X \cong ({}^*X)^*$, it follows that $X^* \cong {}^*X \iff X \cong X^{**}$. Choosing such an isomorphism for each object leads to the following definition.

**Definition 2.11.** A **pivotal category** is a right rigid category $\mathcal{C}$ (with chosen right duals) together with a monoidal natural isomorphism $i\colon 1_{\mathcal{C}} \to -^{**}$, the "pivotal structure". They are also called "sovereign" categories.

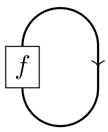
**Lemma 2.12.** A pivotal category is also left rigid, and thus rigid, with the following choice of left dual:

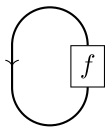
$$ {}^*X := X^* \tag{2.1.8}$$

$$ \widetilde{\mathrm{ev}}_X := \mathrm{ev}_{X^*} \circ (1_{X^*} \otimes i_X) \tag{2.1.9}$$

$$ \widetilde{\mathrm{coev}}_X := \left(i_X^{-1} \otimes 1_{X^*}\right) \circ \mathrm{coev}_{X^*} \tag{2.1.10}$$

**Definition 2.13.** With a pivotal element, left and right traces can be defined.

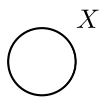
$\mathrm{tr}_R(f) :=$  $= \mathrm{ev}_X \circ (f \otimes 1_{X^*}) \circ \widetilde{\mathrm{coev}}_X = \mathrm{ev}_X \circ \left(\left(f \circ i_X^{-1}\right) \otimes 1_{X^*}\right) \circ \mathrm{coev}_{X^*}$

$\mathrm{tr}_L(f) :=$  $= \widetilde{\mathrm{ev}}_X \circ \left(1_{{}^*X} \otimes f\right) \circ \mathrm{coev}_X = \mathrm{ev}_{{}^*X} \circ \left(1_{{}^*X} \otimes (i_X \circ f)\right) \circ \mathrm{coev}_X$ (2.1.11)

There are pivotal categories for which $\mathrm{tr}_R \neq \mathrm{tr}_L$ for some objects. Spherical categories eliminate this discrepancy.

**Definition 2.14.** A **spherical category** is a pivotal category with $\mathrm{tr}_R = \mathrm{tr}_L =: \mathrm{tr}$ for every object. The pivotal structure of a spherical category is also called a "spherical structure". The **dimension** of an object $X$ is defined as $d(X) := \mathrm{tr}\,(1_X)$.

The diagram for the dimension of an object is a circle. Note that because of sphericality, it is not necessary to specify a direction on the circle.

$d(X) = \mathrm{tr}(1_X) =$ 

(2.1.12)

Note that the dimension of a simple object is known to be nonzero in fusion categories [ENO05]. This follows from the facts that for a simple object $Z$ the spaces $\mathcal{C}(I, Z \otimes Z^*)$ and $\mathcal{C}(Z \otimes Z^*, I)$ have dimension 1, the evaluation and coevaluations are non-zero elements of these spaces, and lemma 2.5.

*Remark* 2.15. The name "spherical" arises from the fact that the diagram of a morphism can be embedded on the 2-sphere, and every isotopy on the sphere amounts to a relation in the category. The additional axiom of a spherical category corresponds to moving a strand "around the back" of the sphere. However, the spherical axiom implies further identities that don't come from isotopies on the sphere.

**Definition 2.16.** The **spherical pairing** of two morphisms $f\colon X \to Y$ and $g\colon Y \to X$ in a spherical category is defined as

$$\langle f, g\rangle := \operatorname{tr}(gf) = \operatorname{tr}(fg) \tag{2.1.13}$$

**Lemma 2.17.** The spherical pairing on a spherical fusion category is nondegenerate.

*Proof.* With the notation from definitions 2.4 and 2.16, write $\Phi^{-1}(f) = \sum_{Z,i} \alpha_{Z,i} \otimes \beta_Z^i$ and $\Phi^{-1}(g) = \sum_{Z',j} \gamma_{Z',j} \otimes \delta_{Z'}^j$, so that $f = \sum_{Z,i} \beta_Z^i \alpha_{Z,i}$ and $g = \sum_{Z',j} \delta_{Z'}^j \gamma_{Z',j}$. Then

$$\langle f, g\rangle = \sum_{Z,i,Z',j} \operatorname{tr}\left(\delta_{Z'}^j \gamma_{Z',j} \beta_Z^i \alpha_{Z,i}\right) = \sum_{Z,i,Z',j} \operatorname{tr}\left(\gamma_{Z',j} \beta_Z^i \alpha_{Z,i} \delta_{Z'}^j\right)$$

But $\gamma_{Z',j}\beta_Z^i$ is a map from $Z$ to $Z'$, and so is non-zero only if $Z = Z'$. In this case, it is equal to $\left(\beta_Z^i, \gamma_{Z,j}\right) 1_Z$, thus one has

$$\langle f, g\rangle = \sum_{Z,i,j} \left(\beta_Z^i, \gamma_{Z,j}\right) \operatorname{tr}\left(\alpha_{Z,i}\delta_Z^j\right) = \sum_{Z,i,j} \left(\beta_Z^i, \gamma_{Z,j}\right) \left(\delta_Z^j, \alpha_{Z,i}\right) d(Z)$$

The fact that this is non-degenerate follows from lemma 2.5 and the fact the dimensions of simple objects are non-zero. □

**Definition 2.18.** A **pivotal** functor $F\colon \mathcal{C} \to \mathcal{D}$ is a strong monoidal functor preserving the pivotal structure up to canonical isomorphisms. More specifically, the following diagram must commute:

$$\begin{array}{ccc} FX & \xrightarrow{i_{FX}} & (FX)^{**} \\ \downarrow{\scriptstyle Fi_X} & & \downarrow{\scriptstyle u_X^*} \\ F(X^{**}) & \xrightarrow{u_{X^*}} & (F(X^*))^* \end{array} \tag{2.1.14}$$

In this diagram, $u$ is the canonical isomorphism from (2.1.5). So pivotal functors preserve the isomorphism between the left dual and the right dual.

**Lemma 2.19.** Pivotal functors preserve traces and therefore quantum dimensions and the spherical pairing. As elements of $\mathbb{C} \cong \mathcal{C}(I_\mathcal{C}, I_\mathcal{C}) \cong \mathcal{D}(I_\mathcal{D}, I_\mathcal{D})$, it follows that $\forall f\colon X \to X$:

$$\operatorname{tr}(f) = \operatorname{tr}(Ff) \tag{2.1.15}$$

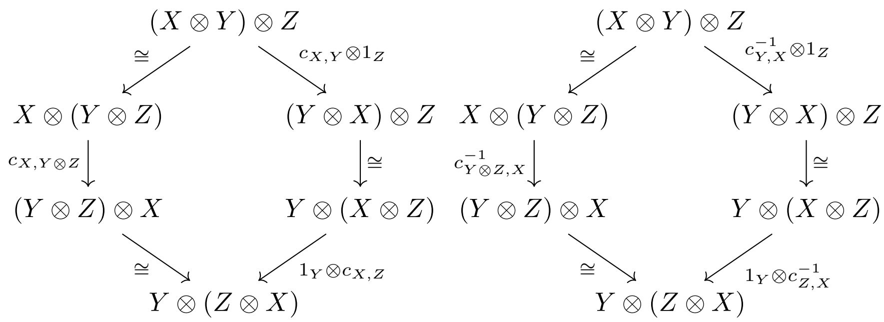
*Proof.* Insert the isomorphism $\mathcal{C}(I_\mathcal{C}, I_\mathcal{C}) \xrightarrow{F} \mathcal{D}(FI_\mathcal{C}, FI_\mathcal{C}) \xrightarrow{F^0 \circ - \circ \left(F^0\right)^{-1}} \mathcal{D}(I_\mathcal{D}, I_\mathcal{D})$ explicitly. As morphisms, it is necessary to prove $F\operatorname{tr}(f) \circ F^0 = F^0 \circ \operatorname{tr}(Ff)$.

$$\begin{aligned} F\operatorname{tr}(f) \circ F^0 &= F\left(\operatorname{ev}_X \circ \left(\left(f \circ i_X^{-1}\right) \otimes 1_{X^*}\right) \circ \operatorname{coev}_{X^*}\right) \circ F^0 \\ &= F\operatorname{ev}_X \circ F^2_{X,X^*} \circ \left(\left(Ff \circ Fi_X^{-1}\right) \otimes 1_{F(X^*)}\right) \circ \left(F^2_{X^{**},X^*}\right)^{-1} \circ F\operatorname{coev}_{X^*} \circ F^0 \\ &= F^0 \circ \operatorname{ev}_{FX} \circ \left(\left(Ff \circ Fi_X^{-1} \circ u_{X^*}^{-1}\right) \otimes u_X\right) \circ \left(F^2_{X^{**},X^*}\right)^{-1} \circ \operatorname{coev}_{F(X^*)} \\ &= F^0 \circ \operatorname{ev}_{FX} \circ \left(\left(Ff \circ i_{FX}^{-1} \circ \left(u_X^*\right)^{-1}\right) \otimes u_X\right) \circ \operatorname{coev}_{F(X^*)} \\ &= F^0 \circ \operatorname{ev}_{FX} \circ \left(\left(Ff \circ i_{FX}^{-1}\right) \otimes 1_{(FX)^*}\right) \circ \operatorname{coev}_{(FX)^*} = F^0 \circ \operatorname{tr}(Ff) \end{aligned}$$

□

### 2.1.5 Braided and ribbon categories

**Definition 2.20.** A **braided monoidal category** (or simply "braided category") is a monoidal category $\mathcal{C}$ with a dinatural isomorphism $c$ (the "braiding") with components $c_{X,Y} : X \otimes Y \to Y \otimes X$ satisfying compatibility axioms with the monoidal product, the hexagon identities:

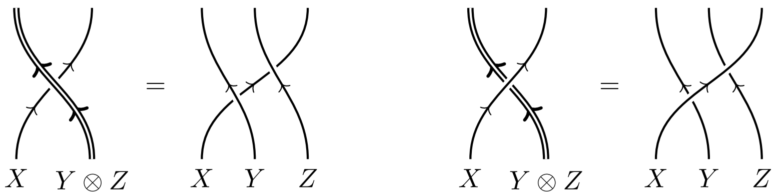



(2.1.16)

As the name suggests, the graphical calculus for braidings consists of strings which can cross each other:


(2.1.17)

The hexagon identities then become




(2.1.18)

**Definition 2.21.** A **ribbon** category is a braided, rigid category $\mathcal{C}$ with a natural isomorphism $\theta \colon 1_{\mathcal{C}} \Rightarrow 1_{\mathcal{C}}$, the "twist", satisfying

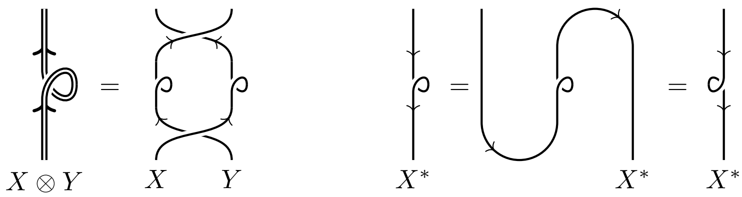
$$\theta_{X\otimes Y} = c_{Y,X} \circ (\theta_Y \otimes \theta_X)\, c_{X,Y} \tag{2.1.19}$$
$$\theta_{X^*} = \theta^*_X \tag{2.1.20}$$

Ribbon categories are also called "tortile" categories.

The graphical representation of the twist is a ribbon that has been twisted by 360 degrees. In knot theory, the lines in a ribbon category are thickened to two-dimensional ribbons in order to express the fact that the twist cannot be undone by an ambient isotopy in three-dimensional space. In two-dimensional diagrams, we can still draw the ribbons as lines and implicitly assume the blackboard framing.

$$\theta_X =$$ (2.1.21)


The axioms for a ribbon category are thus:




(2.1.22)

the last equality introducing the graphical representation for $\theta^*_X$.

**Definition 2.22.** Finitely semisimple ribbon categories are called **premodular**, or **ribbon fusion** categories.

*Remark* 2.23. Ribbon categories have a canonical pivotal structure that is spherical. The spherical condition is a consequence of (2.1.20). As a partial converse, a braided spherical category has a canonical ribbon structure if it is fusion. For more details see [Dri+10, definition 2.29] and the references therein.

### 2.1.6 Symmetric categories

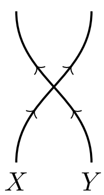
**Definition 2.24.** A braided category is called **symmetric** iff $c_{X,Y} = c^{-1}_{Y,X}$. A finitely semisimple symmetric category is called a **symmetric fusion category**.

*Remarks* 2.25.

- In graphical calculus, the underbraiding and the overbraiding are set equal.
- As a consequence of (2.1.19), a ribbon category is symmetric if the twist is trivial, although there exist symmetric ribbon categories with non-trivial twist.

If the braiding is symmetric, there is no distinction between over- and underbraiding in the diagrammatic calculus:

$$c_{X,Y} = c^{-1}_{X,Y} =$$ (2.1.23)




**Theorem 2.26** (After Deligne, [Del02])**.** In a symmetric fusion category, dimensions of simple objects are integers. If the twist is trivial and all dimensions are positive, there exists a fibre functor to vector spaces, and the symmetric fusion category can be reconstructed as the representations of a finite group.

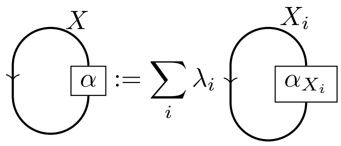
## 2.2 Diagrammatic calculus on spherical fusion categories

**Definition 2.27.** For a fusion category $\mathcal{C}$, let the **fusion algebra** $\mathbb{C}[\mathcal{C}]$ be the complex algebra generated by its objects, modulo isomorphisms and the relations $X \oplus Y = X + Y$ and $X \otimes Y = XY$.

*Remark* 2.28. If $\mathcal{C}$ is braided, $\mathbb{C}[\mathcal{C}]$ is commutative.

By a handy generalisation of notation, closed loops involving only (extra-)natural transformations $\alpha$ are labelled not only with objects, but also with elements of the fusion algebra, in this context called **colours**. The evaluation of a diagram with a linear combination of objects is defined as the sum of the evaluations of the diagrams with the individual objects:

$$X := \sum_i \lambda_i X_i \tag{2.2.1}$$




(2.2.2)

Since braiding and twist are natural transformations, colours can be used in the diagrammatic calculus.

**Definition 2.29** (Graphical calculus for links)**.** Let $L$ be a framed link with a partition of its components into $N$ sets. Given a labelling $(X_1, X_2, \ldots, X_N)$ of the sets with colours from a ribbon fusion category, choose any diagram of the link in the plane, label the link components in each set with the colour of the set and interpret the diagram as a morphism in the category. This amounts for taking identity morphisms for vertical lines, braidings for crossings, (co)evaluations for horizontal lines, twists for framings and composing and tensoring them according to the vertical and horizontal structure of the diagram. This is explained rigorously in [Shu94] and [Sel09].

Note that the choice of diagram for the link doesn't matter as two diagrams only differ by isotopies and Reidemeister moves, which amount to identities (e.g. naturality squares or axioms like the snake identity) in the category. Since a link has no open ends, its resulting morphism will go from $I$ to $I$, it is therefore a complex number. This is denoted $\langle L(X_1, X_2, \ldots, X_N)\rangle$ and called the **evaluation** of the labelled link diagram (not to be confused with the evaluation morphisms $\mathrm{ev}_X$). A link diagram will often be used interchangeably with its evaluation.

**Definition 2.30.** The Kirby colour $\Omega_{\mathcal{C}}$ of a spherical fusion category $\mathcal{C}$ is defined as the sum over the simple objects in $\Lambda_{\mathcal{C}}$ weighted by their dimensions:

$$\Omega_{\mathcal{C}} := \sum_{X \in \Lambda_{\mathcal{C}}} d(X)\, X \tag{2.2.3}$$

Its dimension $d(\Omega_{\mathcal{C}}) = \sum_{X \in \Lambda_{\mathcal{C}}} d(X)^2$ is known as the global dimension of the category. It is always positive, since the field $\mathbb{C}$ has characteristic zero [ENO05].

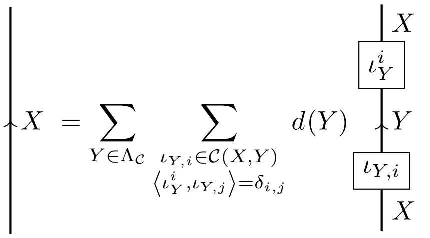
The following two lemmas are well-known, e.g., in [CKY97, section 2].

**Lemma 2.31** (Schur's lemma)**.** Any endomorphism $f \colon X \to X$ of a simple object with non-zero dimension satisfies:

$$f = 1_X \cdot \frac{\mathrm{tr}(f)}{d(X)} \tag{2.2.4}$$

*Proof.* Since $X$ is simple, $\mathcal{C}(X,X) \cong \mathbb{C}$, so every endomorphism is a multiple of the identity. Taking the trace on both sides of the equation $f = \lambda 1_X$ yields the result. □

**Lemma 2.32** (Insertion lemma)**.** For any object $X$ in a spherical fusion category, the Kirby colour can always be inserted in $X$'s identity:




(2.2.5)

Note that the sum over $Y$ ranges over the simple objects. The $\iota_{Y,i}$ form a basis of $\mathcal{C}(X,Y)$ to which the $\iota_Y^j \in \mathcal{C}(Y,X)$ are the dual basis with respect to the spherical pairing $\langle -,- \rangle$ defined in (2.1.13).

*Proof.* The definition of semisimplicity 2.4 implies that

$$1_X = \sum_{Y,i} \beta_Y^i \iota_{Y,i} \tag{2.2.6}$$

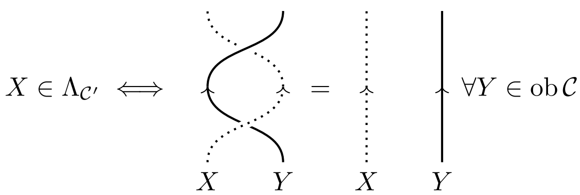
for some $\beta_Y^i \in \mathcal{C}(Y,X)$. Then inserting this equality into $\langle \iota_{Z,j}, 1_X \iota_Z^k \rangle$ shows that $\beta_Y^i = d(Y)\, \iota_Y^i$. □

This is a generalisation of the fact from linear algebra that any vector can be decomposed uniquely into a linear combination of basis vectors.

### 2.2.1 Ribbon fusion categories

This subsection introduces some notation and known lemmas in ribbon fusion categories. These are also known as premodular categories.

**Definition 2.33.** An object $X$ is called **transparent** (or "central") in $\mathcal{C}$ if it braids trivially with any object in $\mathcal{C}$, that is, $c_{Y,X} \circ c_{X,Y} = 1_{X\otimes Y}$. The full symmetric monoidal subcategory $C' \subset \mathcal{C}$ with all transparent objects of $\mathcal{C}$ is called "symmetric centre" or "centraliser" of $\mathcal{C}$, as for example in [Mü03b] or [Dri+10]. The set of equivalence classes of simple transparent objects in $\mathcal{C}$ is then $\Lambda_{\mathcal{C}'}$. Dotted lines represent transparent objects.



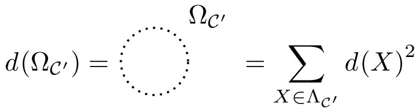

(2.2.7)

**Definition 2.34.** Assume $X$ is a colour $X = \lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_N X_N$ with all $X_i$ simple, further assume that $X_1$ to $X_k$ are transparent and $X_{k+1}$ to $X_N$ are not. Then define the **transparent colour** $X' := \lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_k X_k + 0X_{k+1} + \cdots + 0X_N$.

**Definition 2.35.** A transparent Kirby colour is defined as follows.

$$\Omega_{\mathcal{C}'} = \Omega'_{\mathcal{C}} = \sum_{X \in \Lambda_{\mathcal{C}'}} d(X)\, X \tag{2.2.8}$$

In the same manner, the transparent dimension is defined:




(2.2.9)

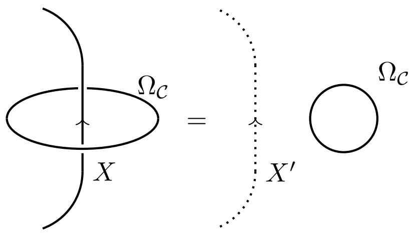
**Definition 2.36.** A category is called **modular**, if it has $\Lambda_{C'} = \{I\}$, $I$ being the monoidal identity.

The transparent dimension of a modular category is 1. Note that the multifusion case, where $I$ is not a simple object, is excluded.

*Remark* 2.37. An object that is not transparent in $\mathcal{C}$ can still be transparent in a subcategory $\mathcal{B} \subset \mathcal{C}$.

The technique of encirclement allows for many elegant and yet powerful calculations and is indispensable when defining invariants derived from ribbon fusion categories and Kirby diagrams. Its power comes from the so-called **killing property**. This is also known as the Lickorish encircling lemma [Lic93], see also [Rob95].

**Lemma 2.38** (Killing property)**.** In a ribbon fusion category, the following holds for any object:

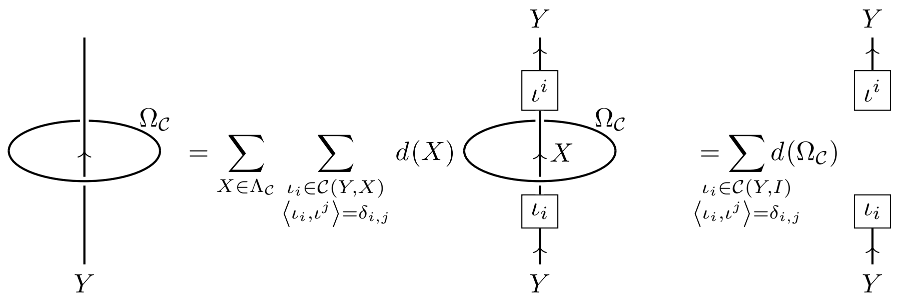



(2.2.10)

Let in particular $X$ be simple. Then $X' = X$ if it is transparent, and 0 otherwise. In the latter case one says that $X$ is "killed off". Note that the circle containing $\Omega_{\mathcal{C}}$ does not need to be oriented since the colour is self-dual.

Combining the Killing property 2.38 and the insertion lemma 2.32 enables us to explicitly write down the morphism of two strands labelled with simple objects encircled with the Kirby colour.

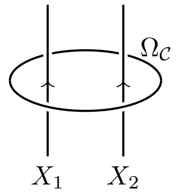
**Lemma 2.39** (Cutting strands)**.** Let $Y$ be an arbitrary object of a modular category $\mathcal{C}$. Then:



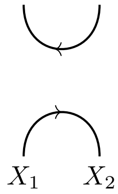

(2.2.11)

The last step uses the fact that in a modular category, $I$ is the only transparent object.

**Lemma 2.40** (Cutting two strands)**.** Let $X_1, X_2$ be simple objects of a modular category $\mathcal{C}$. Then as a special case of the previous lemma:




$$= \delta_{X_1^*, X_2} d(X_1)^{-1} d(\Omega_{\mathcal{C}}) \qquad (2.2.12)$$




To see the prefactors, observe that $\mathcal{C}(X_1 \otimes X_2, I) \cong \mathcal{C}(X_2, X_1^*)$. This is isomorphic to $\mathbb{C}$ if $X_1^* \cong X_2$, and 0 otherwise. If $X_1^* \cong X_2$, then $\mathcal{C}(X_1 \otimes X_2, I)$ is spanned by $\mathrm{ev}_{X_1}$. Since $\widetilde{\mathrm{coev}}_{X_1} \circ \mathrm{ev}_{X_1} = d(X_1)$, the dual basis element must be $\widetilde{\mathrm{coev}}_{X_1} \cdot d(X_1)^{-1}$.

## 2.3 4-Manifolds and Kirby calculus

An extensive treatment of these topics is found in [GS99]. The essential definitions and facts are highlighted here.

### 2.3.1 Handle decompositions

Let $D^k$ denote the closed $k$-disk. The space $D^k \times D^{4-k}$, $k \in \{0, 1, 2, 3, 4\}$, is called a 4-dimensional $k$-handle. All handles have the same underlying topological space, but they differ in the way they are attached to each other. The boundary of a $k$-handle is $\partial \left(D^k \times D^{4-k}\right) = S^{k-1} \times D^{4-k} \cup D^k \times S^{3-k}$, where $S^{-1} = \emptyset$. The first component of the boundary is called the attaching region.

Smooth manifolds admit handle decompositions. A $k$-handle can be attached to a manifold with boundary by embedding its attaching region into the boundary of the manifold. A $k$-handlebody is obtained by attaching a disjoint union of $k$-handles to a $k-1$ handlebody, and is thus a union of 0, 1, ... and $k$-handles. Note that 0-handles have no attaching region, and a 0-handlebody is just a disjoint union of 0-handles, which are $D^4$s. Every $n$-manifold can be decomposed into handles, that is, it is diffeomorphic to an $n$-handlebody.

The handle decomposition is by no means unique. Two handle decompositions of diffeomorphic manifolds are always related by "handle moves", which are either cancellations of a $k$- and a $(k+1)$-handle, or a slide of a $(k+m)$- over a $k$-handle. For a connected manifold it is always possible to arrive at a handle decomposition with exactly one 0-handle by cancelling 0-1 pairs. Similarly, for a closed connected $n$-manifold it is always possible to have exactly one $n$-handle by cancelling $(n-1)$-$n$ pairs.

### 2.3.2 Kirby diagrams and dotted circle notation

For the 2-handlebody of a four-manifold, one can specify the handles and their attaching maps by identifying the boundary of the single 0-handle with $\mathbb{R}^3 \cup \infty$ and drawing pictures of the attaching regions of the 1- and 2-handles. This is explained in [GS99, section 5.1]. A 1-handle amounts to choosing two 3-balls $D^3 \sqcup D^3 \subset \mathbb{R}^3$, which are identified. A 2-handle is an embedding of $D^2 \times S^1$, which is, up to isotopy, a framed embedding of $S^1$, i.e. a framed knot. When part of the 2-handle is attached to a 1-handle, the $S^1$ of the 2-handle will enter one of the 3-balls of the 1-handle and leave the 3-ball with which the former has been identified. The diagram of the attaching regions in $\mathbb{R}^3$ is called a **Kirby diagram**. Some examples can be found in section 6.2.

A theorem ensures that for a closed 4-manifold $M$, specifying the 2-handlebody of a handle decomposition of $M$ determines up to diffeomorphism. This is because there is only one way of adding the 3- and 4-handles. Thus a closed manifold is specified uniquely (up to diffeomorphism) by its Kirby diagram.

The **dotted circle notation** for 1-handles developed by Akbulut is sometimes more convenient. The two balls are connected with a framed interval, or an embedding of $D^2 \times [-1,1]$ and instead of drawing the balls, draw $S^1 \times \{0\} \subset D^2 \times [-1,1]$. A 2-handle running over this 1-handle is then drawn as a continuous line going through this $S^1$. This gives a so-called **special framed link** as a description of the 4-manifold. To distinguish 1- and 2-handles, dots are drawn on the 1-handles. Note that the sublink consisting of only 1-handles is unframed and unlinked, but 2-handles can of course be framed, linked with 1-handles and linked amongst themselves.

This representation of a handle decomposition of a closed, oriented 4-manifold is important in the definition of the invariants. The special framed link diagram $L$ can be labelled using two objects $X$ and $Y$ of a ribbon category. Each dotted link component (1-handle) is labelled with the colour $X$ and each of the remaining components (2-handles) is labelled with $Y$. The labelled link is denoted $L(X,Y)$ and the evaluation of the link diagram $\langle L(X,Y)\rangle$.

### 2.3.3 The fundamental group

A Kirby diagram for a manifold $M^4$ gives rise to a presentation of its fundamental group $\pi_1(M)$. The generators are the 1-handles, while the 2-handles are the relations.

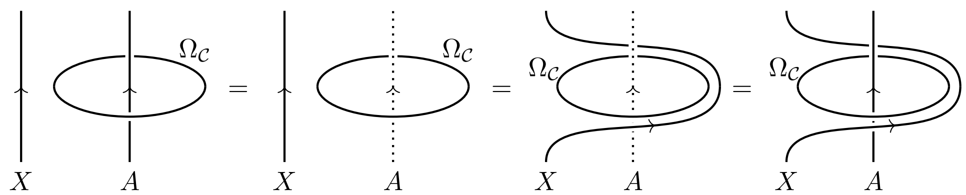
More specifically, choosing a basepoint in the 0-handle, there is a homotopy class of non-contractible curves going through a 1-handle. Attaching a 2-handle gives a way of contracting the $S^1$ on its attaching region. Thus the composition of the curves going through the 1-handles along which the 2-handles is attached can be equated with the contractible curve.

In a Kirby diagram, this will look like this: Each pair of 3-balls is labelled with a generator and its inverse, respectively. For every undotted circle choose an orientation and construct a word of generators by going once along the circle, writing down the generator (or its inverse) when passing through a 3-ball. The resulting word is then a relation of the fundamental group.

# 3 The generalised dichromatic invariant

## 3.1 The generalised sliding property

**Lemma 3.1.** In a ribbon fusion category $\mathcal{C}$, the sliding property due to Lickorish [Lic93] holds.




(3.1.1)

The killing property 2.38 has been used twice.

As the diagrams suggest, the sliding property will later ensure that the invariant doesn't change under handleslides. To label 2-handles differently than 1-handles, it is necessary to generalise the sliding property of lemma 3.1 to ensure invariance under the 2-2-handle slide. The idea will be to label the 2-handles with $F\Omega_\mathcal{C}$ where $F$ is a suitable functor. Then encirclements with $F\Omega_\mathcal{C}$ must also satisfy a sliding property.

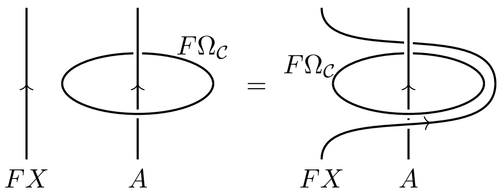
Lemma 2.32, which states that the Kirby colour can be inserted into the identity of any object, can be generalised.

**Lemma 3.2** (Generalised insertion lemma)**.** Let $F\colon \mathcal{C} \to \mathcal{D}$ be a pivotal functor. Then $F\Omega_{\mathcal{C}}$ can be inserted into the identity of $FX$.

*Proof.* Apply $F$ to both sides of equation (2.2.5) in the insertion lemma. Since pivotal functors preserve traces, they also preserve quantum dimensions and dual bases. ☐

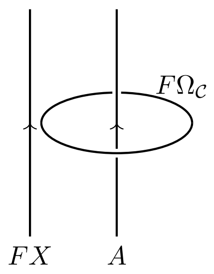
The Lickorish sliding property is generalised here to the dichromatic situation.

**Lemma 3.3** (Generalised sliding property)**.** Let $X \in \operatorname{ob}\mathcal{C}$ and $A \in \operatorname{ob}\mathcal{D}$. Let also $F\colon \mathcal{C} \to \mathcal{D}$ be a pivotal functor from a spherical fusion category to a ribbon fusion category. Then this generalisation of the sliding property holds for all objects $X \in \operatorname{ob}\mathcal{C}, A \in \operatorname{ob}\mathcal{D}$:



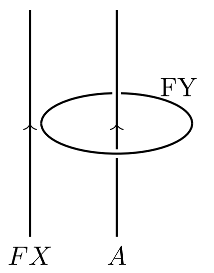
(3.1.2)

*Proof.*



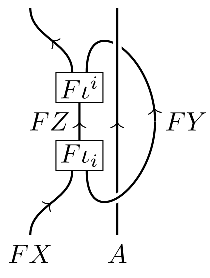
(3.1.3)

$= \sum_{Y \in \Lambda_{\mathcal{C}}} d(Y)$



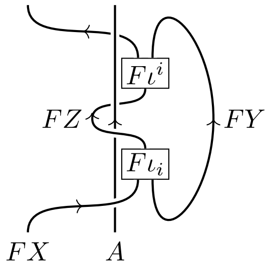

Using the definition of $\Omega_{\mathcal{C}}$. Quantum dimensions are preserved since $F$ is pivotal. (3.1.4)

$= \sum_{\substack{Y,Z \in \Lambda_{\mathcal{C}} \\ \iota_i \in \mathcal{C}(Z, X \otimes Y^*) \\ \langle \iota^i, \iota_j \rangle = \delta_{i,j}}} d(Y)\, d(Z)$



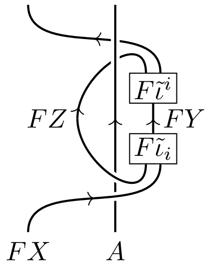

Insertion of $F\Omega_{\mathcal{C}}$, according to lemma 3.2. (3.1.5)

$$= \sum_{\substack{Y,Z\in\Lambda_{\mathcal{C}} \\ \iota_i \in \mathcal{C}(Z, X\otimes Y^*) \\ \langle \iota^i, \iota_j \rangle = \delta_{i,j}}} d(Y)\, d(Z)$$



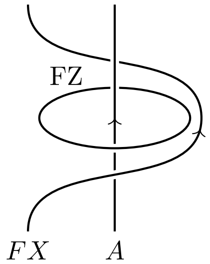

Isotopies of diagrams in the plane and braiding as natural transformation in $\mathcal{D}$. (3.1.6)

$$= \sum_{\substack{Y,Z\in\Lambda_{\mathcal{C}} \\ \tilde\iota_i \in \mathcal{C}(Y, Z^*\otimes X) \\ \langle \tilde\iota^i, \tilde\iota_j \rangle = \delta_{i,j}}} d(Y)\, d(Z)$$



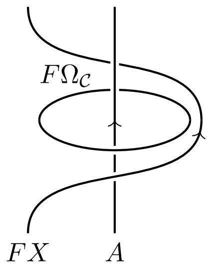

Isomorphism $\iota_i \mapsto \tilde\iota_i$ between $\mathcal{C}(Z, X\otimes Y^*)$ and $\mathcal{C}(Y, Z^*\otimes X)$ given by composition with evaluation and coevaluation. The $\tilde\iota$ form dual bases because of pivotality of $F$ and sphericality of $\mathcal{D}$. (Explained below.) (3.1.7)

$$= \sum_{Z\in\Lambda_{\mathcal{C}}} d(Z)$$



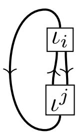

Inverse insertion of $F\Omega_{\mathcal{C}}$. Note that the two $F\Omega_{\mathcal{C}}$'s have swapped roles during the process. (3.1.8)

$$=$$




(3.1.9)

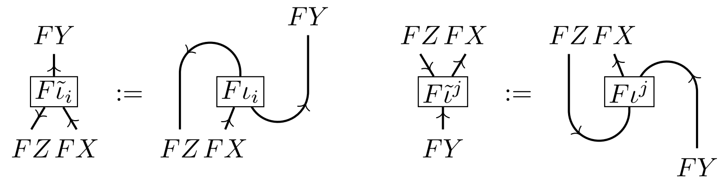
The non-obvious part of the calculation is (3.1.7). The assumption that $\{\iota_i\}$ and $\{\iota^i\}$ form dual bases with respect to the spherical pairing looks like this in graphical calculus:




$$= \delta_{i,j} \tag{3.1.10}$$

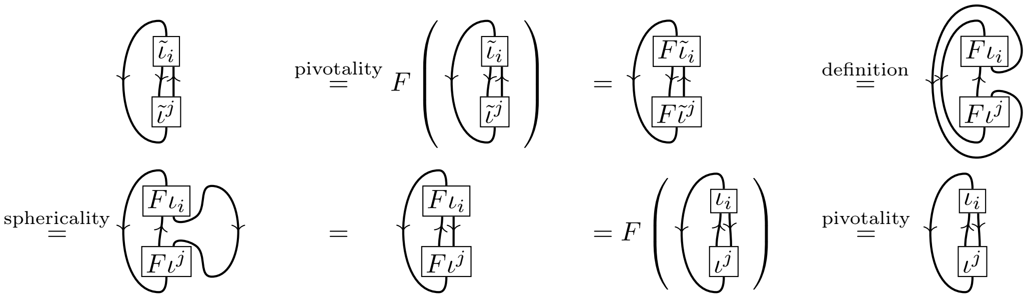
It is necessary to show that this property is also true for $\{\tilde\iota_i\}$ and $\{\tilde\iota^j\}$. After composing the $F\iota_i$ and $F\iota^j$ with evaluations and coevaluations, this again results in $F$ applied to morphisms $\{\tilde\iota_i\}$

and $\{\tilde{\iota}^j\}$:




(3.1.11)

This is true since $F$ is monoidal and therefore preserves duals up to the natural isomorphism $F^2$ which is implicit here. It is necessary to show now that $\{\tilde{\iota}_i\}$ and $\{\tilde{\iota}^j\}$ are dual bases again. But this follows from pivotality of $F$ (preservation of traces) and sphericality of $\mathcal{D}$:




$$= \quad \delta_{i,j} \tag{3.1.12}$$

In words, pivotal functors preserve dual bases (with respect to the spherical pairing). □

*Remark* 3.4. The lemma works as well if the encircling is not an unknot: Braidings and twists are natural transformations and can therefore be pushed past the $F\tilde{\iota}_i$, so they will be passed on to the new encircling morphism and the slid handle. It is remarkable that it is not necessary to demand $\mathcal{C}$ is ribbon, neither that $F$ is braided or ribbon. In fact, this lemma stems from a better-known sliding lemma in spherical categories used for example in understanding the Hilbert spaces assigned to surfaces in the Turaev-Viro-TQFT [Kir11, Corollary 3.5].

For $F$ the identity of a ribbon fusion category, the sliding lemma would have followed directly from the killing property, as demonstrated in 3.1. But if $F$ is not the identity, it is unclear whether there is an analogue to the killing property.

## 3.2 The definition

The definition of the generalised dichromatic invariant can now be given.

**Definition 3.5.** Assume the following:

- Let $\mathcal{C}$ be a spherical fusion category.
- Let $\mathcal{D}$ be a ribbon fusion (premodular) category with trivial twist on all transparent objects.
- Let $F\colon \mathcal{C} \to \mathcal{D}$ be a pivotal functor.
- Let $L$ be the special framed link obtained from a handlebody decomposition of a smooth, oriented, closed 4-manifold $M$.

Then the **generalised dichromatic invariant** of $L$ associated with $F$ is defined as:

$$I_F(L) \colon = \frac{\langle L\,(\Omega_{\mathcal{D}}, F\Omega_{\mathcal{C}})\rangle}{d(\Omega_{\mathcal{C}})^{h_2-h_1}\,\big(d(\Omega_{\mathcal{D}})\,d\big((F\Omega_{\mathcal{C}})'\big)\big)^{h_1}} \tag{3.2.1}$$

Here, $h_i$ is the number of $i$-handles of the handle decomposition, or, the number of components in the first, respective, second set of the special framed link. $\langle L\,(\Omega_{\mathcal{D}}, F\Omega_{\mathcal{C}})\rangle$ is the evaluation of the special framed link diagram as a morphism in $\mathcal{D}$, as described in section 2.3.2. $(F\Omega_{\mathcal{C}})'$ is the transparent part of $F\Omega_{\mathcal{C}}$, see definition 2.34.

*Remark* 3.6. It might be counter-intuitive that the unknotted, unframed, unlinked 1-handles are being labelled by $\Omega_{\mathcal{D}}$, while the 2-handles are labelled by $F\Omega_{\mathcal{C}}$, but $\mathcal{D}$ is the ribbon category (which has algebraic counterparts of knots and framings) and $\mathcal{C}$ is only spherical. But this is indeed a valid definition, while a functor in the other direction does not obviously lead to a similar invariant.

Note also that $F\Omega_{\mathcal{D}}$ does not depend on the monoidal coherence $F^2$ of $F$. Two functors with different $F^2$ will give the same invariant. Furthermore, any two isomorphic functors will also yield the same invariant.

In the rest of the article the conditions in the definition will be assumed, unless stated otherwise.

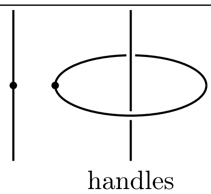
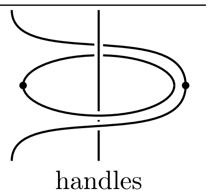
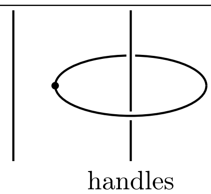
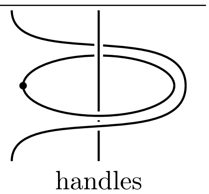
## 3.3 Proof of invariance

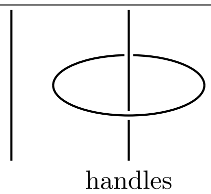
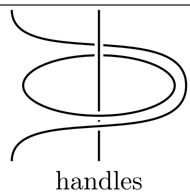
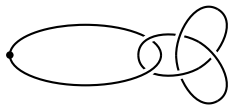
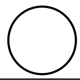
**Lemma 3.7** (Multiplicativity under disjoint union)**.** For two links $L_1$ and $L_2$, $I_F$ is multiplicative under disjoint union $\sqcup$:

$$I_F(L_1 \sqcup L_2) = I_F(L_1) \cdot I_F(L_2) \tag{3.3.1}$$

*Proof.* Evaluation of the graphical calculus is multiplicative under disjoint union: A link corresponds to an endomorphism of $\mathbb{C}$, so two links correspond to an endomorphism of $\mathbb{C} \otimes \mathbb{C}$. The evaluation is a monoidal functor with coherence isomorphism $- \cdot - \colon \mathbb{C} \otimes \mathbb{C} \to \mathbb{C}$, so the numerator of $I_F$ is multiplicative. Obviously, $h_i(L_1 \sqcup L_2) = h_i(L_1) + h_i(L_2)$, so the denominator is multiplicative as well. □

Recall that two links assigned to different handle decompositions of a manifold can be transformed into each other by a series of handle slides and handle cancellations, as described for example in [GS99]. The relevant moves are:

| Handle move | before | after |
| --- | --- | --- |
| 1-1-handle slide |  handles |  handles |
| 2-1-handle slide |  handles |  handles |

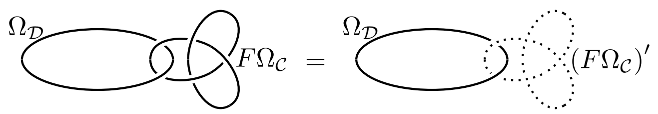
| | | |
|---|---|---|
| 2-2-handle slide |  |  |
| 1-2-handle cancellation |  | (empty) |
| 2-3-handle cancellation |  | (empty) |

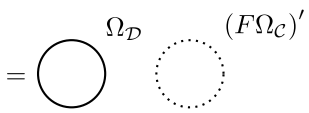
As usual, a dot denotes a 1-handle.

**Theorem 3.8** (Independence of handlebody decomposition)**.** The generalised dichromatic invariant is independent of the handlebody decomposition and is thus an invariant of smooth 4-manifolds.

*Proof.* It is only necessary to check invariance of $I_F$ under each of these moves in order to prove the theorem.

- Invariance under the 1-1-handle slide and the 2-1-handle slide are ensured by the sliding property 3.1. Since 1- and 2-handles are labelled with objects in $\mathcal{D}$, they can slide over a 1-handle which is labelled with $\Omega_{\mathcal{D}}$.
- Invariance under the 2-2-handle slide is ensured by the generalised sliding property 3.3. Every object in the image of $F$ can slide over $F\Omega_{\mathcal{C}}$, so since 2-handles are labelled with $F\Omega_{\mathcal{C}}$, they can slide over each other.
- The 1-2-handle cancellation leaves $I_F$ invariant because of its normalisation. Assume that there is a linked pair of a 1-handle and a 2-handle that is not linked to the rest of the diagram. Then it will be shown that $I_F$ does not change if the pair is removed from the diagram. The 2-handle can be knotted, as is illustrated here with a trefoil knot. Since $I_F$ is multiplicative under disjoint union of link diagrams, it only remains to show that the invariant of the pair of handles evaluates to 1. The numerator is just the evaluation of the graphical calculus and thus results in:




$$= d(\Omega_{\mathcal{D}})\, d\big((F\Omega_{\mathcal{C}})'\big) \tag{3.3.2}$$

The number of 1-handles and 2-handles are both 1, so the denominator the same as the above expression, thus the invariant is 1.

- Invariance under the 2-3-handle cancellation is even easier to show: Since there is a canonical way to attach 3- and 4-handles, they don't appear in the link picture. A 2-3-handle cancellation thus amounts to the removal of an unlinked, unknotted 2-handle. By a similar argument as before, one can evaluate the invariant on the link diagram of such a 2-handle and find that it is 1 as well.

□

*Remark* 3.9. Pivotality of $F$ is essential for the invariance of $I_F$. As an easy counterexample, take the representations of $\mathbb{Z}_2$ and equip the sign representation $\sigma$ with dimension $-1$. (This is essentially the category of super vector spaces with trivial braiding and twist.) There is an obvious forgetful strong monoidal functor $U$ to vector spaces sending both simple objects to $\mathbb{C}$. One finds that the evaluation of the (undotted) unknot is

$$\begin{aligned} d\left(U\Omega_{\mathrm{Rep}(\mathbb{Z}_2)}\right) &= \sum_{X\in\Lambda_{\mathrm{Rep}(\mathbb{Z}_2)}} d(X)\, d(UX) \\ &= 1\cdot 1 + (-1)\cdot 1 = 0 \end{aligned}$$

However, the corresponding manifold is $S^4$ and the empty diagram evaluates to 1.

## 3.4 Multiplicativity under connected sum

The following lemma is known.

**Lemma 3.10.** Assume $I$ is an invariant of oriented, closed four-manifolds that is multiplicative under connected sum on simply-connected manifolds. Furthermore, assume that $I\left(\mathbb{CP}^2\right)$ and $I\left(\overline{\mathbb{CP}^2}\right)$ is invertible. Then $I$ is given on a simply-connected four-manifold $M$ by

$$I(M) = \left(I\left(\mathbb{CP}^2\right) I\left(\overline{\mathbb{CP}^2}\right)\right)^{-1+\chi(M)/2} \left(\frac{I\left(\mathbb{CP}^2\right)}{I\left(\overline{\mathbb{CP}^2}\right)}\right)^{\sigma(M)/2}. \tag{3.4.1}$$

$\chi$ and $\sigma$ are Euler characteristic and signature, respectively.

*Proof.* The first, and by Poincaré duality third, homologies of $M$ are trivial, so the Euler characteristic $\chi(M)$ is equal to $2 + b_2(M)$, where $b_2(M) = b_2^+(M) + b_2^-(M)$ is the rank of the second homology and $b_2^\pm(M)$ are the maximal dimension of the subspaces on which the intersection form is positive or negative. Since the signature is $\sigma(M) = b_2^+(M) - b_2^-(M)$, then it follows that $b_2^\pm(M) = (\chi(M) \pm \sigma(M))/2 - 1$.

But simply-connected manifolds stably decompose into $\mathbb{CP}^2$ and $\overline{\mathbb{CP}^2}$ under direct sum:

$$\exists m \in \mathbb{N}: \quad M\# \left(\overset{m}{\#} \left(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}\right)\right) \cong \left(\overset{m+b_2^+(M)}{\#} \mathbb{CP}^2\right) \# \left(\overset{m+b_2^-(M)}{\#} \overline{\mathbb{CP}^2}\right) \tag{3.4.2}$$

Then $I(M) = I\left(\mathbb{CP}^2\right)^{b_2^+(M)} I\left(\overline{\mathbb{CP}^2}\right)^{b_2^-(M)}$. □

**Lemma 3.11.** The generalised dichromatic invariant is invertible on $\mathbb{CP}^2$ and $\overline{\mathbb{CP}^2}$, thus the previous lemma applies to it.

*Proof.* Multiplicativity under connected sum has already been shown in 3.7. $I\left(\mathbb{CP}^2\right) \cdot I\left(\overline{\mathbb{CP}^2}\right)$ is invertible because $\mathcal{C}'$ has trivial twist. This is explained in [Pet08, proposition 1.2]. □

## 3.5 Examples: Petit's dichromatic invariant and Broda's invariants

Broda defined two invariants of four-manifolds using the category of tilting modules for $U_q sl(2)$ at a root of unity [Bro93; Rob95]. The original invariant, called here the **Broda invariant**, labelled both one- and two-handles with simple objects in this category (the "spins"), whereas the **refined Broda invariant** labelled two-handles with just the even spins. The Broda invariants were investigated by Roberts [Rob95; Rob97], who showed that the Broda invariant depends on the signature of the four-manifold whereas the refined Broda invariant detects also the first Betti number with $\mathbb{Z}_2$ coefficients and whether or not the second Stiefel-Whitney class is zero (i.e., whether the manifold is spin).

Generalising Broda's constructions to other ribbon fusion categories leads to the following two classes of examples.

*Example* 3.12. As noted by Petit [Pet08, remark 4.4], for the identity functor $\mathbb{1}_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}$ one recovers, up to a factor depending on the Euler characteristic, a "generalised" Broda invariant for a ribbon fusion category $\mathcal{D}$ satisfying the conditions of definition 3.5. Petit shows that this invariant depends only on the signature (and Euler characteristic) of the four-manifold.

*Example* 3.13. Let $\mathcal{C}$ and $\mathcal{D}$ be ribbon fusion categories. $\mathcal{D}$ is not required to be modular. For a full ribbon inclusion functor $F \colon \mathcal{C} \hookrightarrow \mathcal{D}$, a generalisation of the refined Broda invariant called **Petit's dichromatic invariant** is recovered, again up to the Euler characteristic $\chi(M)$. The original definition is the third invariant in [Pet08], which he denotes by $I_0$. Note that his other notation is also subtly different: His $\mathcal{C}'$ is a subcategory, not the symmetric centre. Also, his notation for categorical dimensions is different from this presentation. Redefining his symbols in our notation gives $\Delta_{\mathcal{C}} := d(\Omega_{\mathcal{C}})$ and $\Delta_{\mathcal{D},\mathcal{C}} := d\big((F\Omega_{\mathcal{C}})'\big)$.

Note that the numerators of $I_F$ and $I_0$ do not differ, but the normalisations do. Therefore to compare the normalisation of invariants, their ratio is calculated using a handle decomposition with exactly one 0-handle and 4-handle. Since $M$ is closed, the nullity of the linking matrix of the link diagram equals $h_3$, the number of 3-handles. With this, and $h_1 + h_2$ being the number of components of the diagram, the ratio of invariants is

$$\begin{aligned}\frac{I_F(M)}{I_0(M)} &= \frac{d(\Omega_{\mathcal{C}})^{h_3} \cdot \big(d(\Omega_{\mathcal{D}})\, d\big((F\Omega_{\mathcal{C}})'\big)\big)^{(h_1+h_2-h_3)/2}}{d(\Omega_{\mathcal{C}})^{h_2-h_1} \cdot \big(d(\Omega_{\mathcal{D}})\, d\big((F\Omega_{\mathcal{C}})'\big)\big)^{h_1}} \\ &= \left(\frac{\sqrt{d(\Omega_{\mathcal{D}})\, d\big((F\Omega_{\mathcal{C}})'\big)}}{d(\Omega_{\mathcal{C}})}\right)^{\chi(M)-2} \qquad (3.5.1)\end{aligned}$$

*Remark* 3.14. Whenever a full inclusion into a ribbon category is encountered, it will be assumed that the subcategory inherits braiding and ribbon structures from the bigger category. Also it will be assumed that the canonical pivotal structure is chosen on both sides, which is then automatically preserved.

*Remark* 3.15. Petit called his invariant "dichromatic" since the special framed link arising from the handle decomposition is being labelled with two different Kirby colours. The invariant presented here uses two different colours as well so it seems appropriate to keep the name dichromatic but point out that the framework is somewhat more general.

# 4 Simplification of the invariant

Here it is shown that a general argument allows the generalised dichromatic invariant to be simplified in many cases.

**Proposition 4.1.** Let $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C} \xrightarrow{H} \mathcal{D}$ be a chain of pivotal functors on spherical fusion categories. Let furthermore $\mathcal{C} \xrightarrow{H} \mathcal{D}$ be ribbon, and let $\mathcal{C}'$ and $\mathcal{D}'$ have trivial twist. Assume these three conditions on $F$ and $H$:

$$F\Omega_{\mathcal{A}} = n\Omega_{\mathcal{B}}$$
$$H\Omega_{\mathcal{C}} = m\Omega_{\mathcal{D}}$$
$$H\left((G\Omega_{\mathcal{B}})'\right) = (HG\Omega_{\mathcal{B}})'$$

Then $I_{HGF} = I_G$.

*Proof.* Note that the values of $m$ and $n$ can be inferred by taking the dimensions on each side of the conditions. Let $L$ be a special framed link for the 4-manifold $M$.

$$\begin{aligned}\langle L\left(\Omega_{\mathcal{D}}, HGF\Omega_{\mathcal{A}}\right)\rangle &= \langle L\left(H\Omega_{\mathcal{C}}, HG\Omega_{\mathcal{B}}\right)\rangle \cdot m^{-h_1} n^{h_2} \\ &= \langle L\left(\Omega_{\mathcal{C}}, G\Omega_{\mathcal{B}}\right)\rangle \cdot m^{-h_1} n^{h_2}\end{aligned}$$

This used that $H$ is ribbon, and the first two assumptions. Using pivotality of various functors, and all three assumptions, results in the following equations:

$$d(\Omega_{\mathcal{A}}) = n \cdot d(\Omega_{\mathcal{B}})$$
$$d(\Omega_{\mathcal{D}}) = m^{-1} \cdot d(\Omega_{\mathcal{C}})$$
$$d\big((HGF\Omega_{\mathcal{A}})'\big) = n \cdot d\big((HG\Omega_{\mathcal{B}})'\big) = n \cdot d\big(H\left(G\Omega_{\mathcal{B}}\right)'\big) = n \cdot d\big((G\Omega_{\mathcal{B}})'\big)$$

It is easy to see now that all factors of $n$ and $m$ cancel. □

In the following, it will be clear that there is an abundance of functors satisfying these conditions, allowing a simplification of the generalised dichromatic invariant in many cases. Examples include cases where either $H$ or $F$ is the identity functor.

## 4.1 Simplification for unitary fusion categories

One case, in which the generalised dichromatic invariant simplifies to Petit's dichromatic invariant is the case of unitary categories, which are certain non-degenerate dagger $\mathbb{C}$-linear categories. The unitary condition is important in mathematical physics, and many examples are known. The theory of unitary fusion categories is well developed, and many important properties are found in the literature, e.g. [Dri+10]. Instead of giving a self-contained introduction, the relevant known facts are listed.

- A dagger fusion category with a rigid structure has a canonical spherical structure (see [Sel09, Lemma 7.5] defined by the dagger structure and the rigid structure.
- A unitary functor, or dagger functor, is a functor that preserves the dagger structure. Since any strong monoidal functor preserves duals, any unitary strong monoidal functor preserves the canonical spherical structure. More specifically, it is pivotal.

**Definition 4.2.** A strong monoidal functor of fusion categories $F\colon \mathcal{C} \to \mathcal{D}$ is called **dominant** if for any object $Y \in \operatorname{ob}\mathcal{D}$ there exists an object $X \in \operatorname{ob}\mathcal{C}$ such that $Y$ is a subobject of $FX$. In [ENO05] these are also known as "surjective functors".

**Lemma 4.3.** Let $F\colon \mathcal{C} \to \mathcal{D}$ be a dominant unitary functor of unitary fusion categories. Let furthermore both categories have the canonical spherical structure coming from the unitary structure. Then the following holds:

$$F(\Omega_{\mathcal{C}}) = \frac{d(\Omega_{\mathcal{C}})}{d(\Omega_{\mathcal{D}})}\Omega_{\mathcal{D}} \tag{4.1.1}$$

*Proof.* An analogous equation holds for the Frobenius-Perron dimensions [ENO05, proposition 8.8]. In unitary fusion categories with the canonical spherical structure Frobenius-Perron dimensions and categorical dimensions coincide. □

**Lemma 4.4.** Let $F\colon \mathcal{C} \to \mathcal{D}$ be a strong monoidal functor of fusion categories. Then it factors as a dominant functor followed by a full inclusion.

*Proof.* First, define the image category $\operatorname{Im} F$ [Dri+10, Definition 2.1]. Its objects are all objects of $\mathcal{D}$ that are isomorphic to a subobject of $FX$, where $X$ is any object of $\mathcal{C}$. The morphisms of $\operatorname{Im} F$ are such that it is a full fusion subcategory of $\mathcal{D}$. By construction, $F$ factors through $\operatorname{Im} F$, and $F$ restricted to $\operatorname{Im} F$ is dominant. □

**Corollary 4.5.** Let $F\colon \mathcal{C} \to \mathcal{D}$ be a strong monoidal unitary functor of unitary fusion categories. Then $I_F$ is equal to Petit's dichromatic invariant, multiplied by a factor involving the Euler characteristic.

*Proof.* Use the previous lemma to decompose $F$ into a dominant functor and a full inclusion. By the lemma before, the dominant part satisfies the conditions of 4.1, leaving the full inclusion. The fusion subcategory inherits the pivotal structure from $\mathcal{D}$. An invariant from a full inclusion is a case of Petit's dichromatic invariant, as explained in 3.13. □

## 4.2 Modularisation

This subsection considers examples that will be compared to the Crane-Yetter invariant in section 6.

**Definition 4.6.** A ribbon fusion category $\mathcal{D}$ is called **modularisable** if its symmetric centre $\mathcal{D}'$ has trivial twist and dimensions in $\mathbb{N}$. For modularisable categories, there exists an inclusion $H\colon \mathcal{D} \hookrightarrow \widetilde{\mathcal{D}}$ with $\widetilde{\mathcal{D}}$ modular, called the **modularisation** (also "deequivariantisation") of $\mathcal{D}$. Some standard references are [Bru00] or [Mü00].

*Remarks* 4.7.
- $H$ is usually not full.
- The name "deequivariantisation" comes from thinking of $\mathcal{D}'$ as the representations of some finite group. $H$ restricted to $\mathcal{D}'$ then plays the role of a fibre functor, while not disturbing the nontransparent objects. $\widetilde{\mathcal{D}}$ has the same objects as $\mathcal{D}$, but additional isomorphisms from any transparent object to a direct sum of $I$s.
- For any symmetric fusion category, one can choose the trivial twist $\theta_X = 1_X$. With the corresponding pivotal structure, the categorical dimensions of objects are then in $\mathbb{Z}$. Alternatively, one can choose a pivotal structure with categorical dimensions in $\mathbb{N}$, but then the twist will usually not be trivial. To adhere to the conditions in 3.5, the trivial twist will always be chosen in the following.

**Proposition 4.8.** Let $F\colon \mathcal{C} \to \mathcal{D}$ be pivotal with $\mathcal{D}$ modularisable. Such a functor satisfies the conditions of our invariant 3.5. Let $H\colon \mathcal{D} \to \widetilde{\mathcal{D}}$ be the modularisation functor. Then $I_F = I_{H\circ F}$.

*Proof.* In [Bru00, proposition 3.7] it is stated that $H\Omega_{\mathcal{D}} \cong d(\Omega_{\mathcal{D}'})\,\Omega_{\widetilde{\mathcal{D}}}$. It is easy to check that $\widetilde{\mathcal{D}}(I, H(X')) = \widetilde{\mathcal{D}}(I, (HX)')$ follows from the original definition, and therefore $H\left((F\Omega_{\mathcal{C}})'\right) = (HF\Omega_{\mathcal{C}})'$ since both sides are multiples of $I$. Thus, 4.1 can be applied. □

Intuitively, the transparent objects on the 1-handles can be removed and don't contribute to the invariant. The modularisation $H$ makes this explicit by sending all objects in $\mathcal{D}'$ to multiples of $I$.

One can make use of this fact by noting that many generalised dichromatic invariants are equal to an invariant arising from a functor into a modular category. It is necessary to demand all dimensions of simple objects in $\mathcal{D}'$ to be positive, but this is the sole restriction. In section 5, it will be shown that invariants with the target category modular can be expressed in terms of a state sum and therefore extend to topological quantum field theories.

*Remark* 4.9. The modularisation $H$ is not a full inclusion if the source $\mathcal{D}$ is not modular (and the identity otherwise). Therefore, the composition $H \circ F$ will usually not be full either, even if $F$ is. However, if the categories in question are unitary, 4.5 can be used to reduce the functor to a full inclusion.

**Corollary 4.10.** Let $F\colon \mathcal{C} \to \mathcal{D}$ be a strong monoidal unitary functor of unitary fusion categories. Let also $\mathcal{D}$ be modularisable. Then the generalised dichromatic invariant $I_F$ is equal to a case of Petit's dichromatic invariant (i.e. coming from a full inclusion) where the target category is modular.

*Proof.* From 4.8, $I_{H\circ F} = I_F$, where $H$ is the modularisation. Therefore 4.5 can be applied to $H \circ F$. □

## 4.3 Cutting strands

If the target category $\mathcal{D}$ of the pivotal functor is modular, each 1-handle is labelled by $\Omega_{\mathcal{D}}$ one can cut the strands of the 2-handles going through it, using lemma 2.39. This is the algebraic analogue of reverting from Akbulut's dotted handle notation to Kirby's original notation for handle decompositions where each 1-handle is represented by a pair of $D^3$s. There is now a simpler definition of the generalised dichromatic invariant, which is obtained by cutting the strands through the 1-handles.

**Definition 4.11.** Let $K$ be a Kirby diagram for a handle decomposition of a smooth, closed 4-manifold $M$. Choose orientations on the $S^1$ of the attaching boundary of each 2-handle, and a choice of $+$ and $-$ sign on the respective 3-balls for each 1-handle.

1. An **object labelling** is a map $X$ from the set of 2-handles to the set of simple objects in $\mathcal{C}$.

2. Now, for every 1-handle with 2-handles $i \in \{1, \ldots N\}$ entering or leaving the ball labelled with $+$, dual bases for the morphism spaces $\mathcal{D}(FX_1 \otimes FX_2 \otimes \cdots \otimes FX_N, I)$ and $\mathcal{D}(I, FX_1 \otimes FX_2 \otimes \cdots \otimes FX_N)$ are chosen, the objects on leaving 2-handles are dualised.

   A **morphism labelling** for a given object labelling is a choice of basis morphism for the $+$ ball of every 1-handle, and the corresponding dual morphism on the ball labelled with $-$.

3. For a given object and morphism labelling, the evaluation of the labelling is the evaluation of the labelled diagram as a morphism in $\mathcal{D}(I, I) \cong \mathbb{C}$.

4. The evaluation $\langle K(F) \rangle$ of the Kirby diagram $K$ is the sum of evaluations over all labellings.

**Proposition 4.12.** Let $K$ be a Kirby diagram for a handle decomposition of a smooth, closed 4-manifold $M$. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a pivotal functor from a spherical fusion category to a modular category, and let $n$ be the multiplicity of $I$ in $F\Omega_{\mathcal{C}}$. Then the generalised dichromatic invariant is:

$$I_F(M) = \frac{\langle K(F) \rangle}{d(\Omega_{\mathcal{C}})^{h_2 - h_1} \, n^{h_1}} \tag{4.3.1}$$

*Proof.* Application of lemma 2.39 to the labelled special framed link $L$ shows that $\langle L(\Omega_{\mathcal{D}}, F\Omega_{\mathcal{C}}) \rangle = d(\Omega_{\mathcal{D}})^{h_1} \langle K(F) \rangle$. Since $\mathcal{D}$ is modular, $d\big((F\Omega_{\mathcal{C}})'\big) = d(nI) = n$ and the result follows. □

This proposition can be used as an alternative definition of the invariant. However to prove invariance under all handle slides, it is more convenient to refer to the original definition 3.5.

# 5 The state sum model

The Crane-Yetter invariant is defined using a state sum model on a triangulation of the four-manifold [CKY97]. However, the Crane-Yetter invariant was not presented as a state sum model in section 1.1. This is possible using the reformulation of the original definition due to Roberts, as presented in [Rob95, section 4.3]. He showed that for modular categories, the Crane-Yetter state sum $CY$ is equal to the Broda invariant $B$ up to a normalisation involving the Euler characteristic [Rob95].

The original definition of $CY$ can be recovered from $B$ easily through a process called "chain mail", which will be described in the following. This is not true for nonmodular $\mathcal{C}$: As will be shown in the next section, $CY$ and $B$ indeed differ in this case. The nonmodular Crane-Yetter invariant arises from Petit's dichromatic invariant and does not depend only on the signature and Euler characteristic, but also at least on the fundamental group.

Previously, it wasn't known how to derive the nonmodular Crane-Yetter invariant from a handle picture in terms of a chain mail process. With the generalised dichromatic invariant, it is possible to do so. Through chain mail one can recover a state sum description of the generalised dichromatic invariant $I_F$, whenever $F\colon \mathcal{C} \to \mathcal{D}$ such that $\mathcal{D}$ is modular. So the generalised dichromatic invariant has a purely combinatorial description in term of triangulations in that case. The nonmodular Crane-Yetter invariant will turn out to be a special case.

In general, the state sum model will be useful to understand the physical interpretation of a particular model, while the handle picture is very convenient for calculations.

## 5.1 The chain mail process and the generalised 15-j symbol

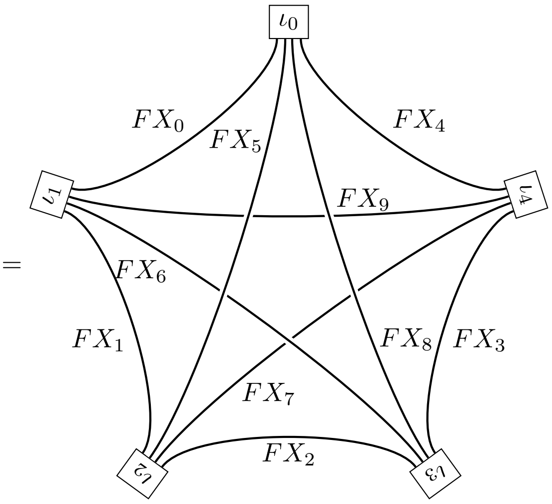
Given an 4-dimensional manifold $M$ with triangulation $\Delta$, there is always an unframed, unknotted handle decomposition via the following process: Replace the triangulation by its dual complex, i.e. 4-simplices $s \in \Delta_4$ by vertices, tetrahedra $t \in \Delta_3$ by edges, triangles $\tau \in \Delta_2$ by polygons and in general $(4-k)$-simplices by $k$-cells. A $k$-cell will then have a valency (the number of adjacent $(k+1)$-cells) of $5-k$, coming from the number of faces of the original simplex.

Then consider the handle decomposition arising from a thickening of this dual complex. This handle decomposition has $h_0$ 0-handles, where $h_0$ is then the number of 4-simplices in the triangulation, $\Delta_4$. To work with Kirby diagrams, a decomposition with only one 0-handle is needed. In [Rob95, section 4.3], Roberts shows that this amounts to normalising by $I_F(S^1 \times S^3)^{1-h_0} = d(\Omega_{\mathcal{C}})^{1-h_0}$.

Inserting morphism boxes for the two $D^3$s at every 1-handle arising from a tetrahedron disconnects the whole link diagram into pentagram-shaped subdiagrams for every 4-simplex.

To arrive at the pentagram shape, first realise that the boundary of a 4-simplex is $S^3 \cong \mathbb{R}^3 \cup \{\infty\}$. This is the boundary of the 0-handle to which 1-handles and 2-handles are attached. Visualise the triangulation of the boundary by arranging four vertices of the 4-simplex as a tetrahedron around the origin and putting the remaining vertex at infinity. Connecting the first four vertices to the vertex at infinity gives the remaining four tetrahedra. Now draw one copy of $D^3$ for each tetrahedron (the other copy belonging to the neighbouring 4-simplex) and connect each pair of $D^3$ with lines from the triangles as 2-handles. Project this subdiagram onto the plane, for every 4-simplex, and apply definition 4.12. After applying an arbitrary isotopy in the plane, the evaluation of such a subdiagram labelled with objects $X_i \in \operatorname{ob}\mathcal{C}$ and morphisms $\iota_i$, $i \in \{0, \ldots, 4\}$, $j \in \{0, \ldots, 9\}$ in $\mathcal{D}$ is:

 $(FX_i, \iota_i) :=$  $(FX_0, \ldots, FX_9, \iota_0, \ldots, \iota_4)$

$:=$  (5.1.1)


The over- and under-braidings follow the convention of Roberts. It involves a "splitting convention" to arrive at a correct blackboard framing, see [Rob95, Fig. 17].

The diagram does not yet correspond to a morphism. To evaluate it in terms of diagrammatic calculus of the ribbon category $\mathcal{D}$, one has to orient the lines upwards or downwards and insert evaluations and coevaluations as needed, in order to specify where an object or its dual is the source or the target of a morphism. To arrive at such a choice, fix a total ordering of the vertices. This ordering induces an orientation on the tetrahedra. Each tetrahedron occurs as the face of two 4-simplices, which are oriented as submanifolds of $M$, and the tetrahedron inherits two opposite orientations from each of them. Since a tetrahedron corresponds to a 1-handle, the $+$ and $-$ signs need to be distributed onto the attaching $D^3$s. Put the $+$ sign on the $D^3$ attaching to the 4-simplex from which the tetrahedron inherits the orientation agreeing with the ordering of the vertices. Consequently, its morphism is $\iota : X_{i_1} \otimes \cdots \otimes X_{i_4} \to I$, while the morphism of the other $D^3$ goes in the other direction.

## 5.2 The state sum

Since the whole diagram is a disconnected sum of diagrams of the above shape, its evaluation will be a product of  -quantities. Remembering the definition of the $\Omega$ in terms of evaluating the diagram as a sum over simple objects, it is possible to define a state sum formula for $I_F$. The $X_i$ in the definition of  are then summands of $F\Omega_{\mathcal{C}}$, which was labelling the 2-handles. The $\iota_i$ label the $D^3$s of a 1-handle. The invariant $I_F$ will then be a big sum over the summands of all these copies of $F\Omega_{\mathcal{C}}$ and the dual morphism bases.

**Definition 5.1.** An $F$-object labelling of the triangulation $\Delta$ is a function

$$X : \Delta_2 \to \Lambda_{\mathcal{C}} \tag{5.2.1}$$

For a given $F$-object labelling and a total ordering of the vertices $\Delta_0$ fix bases of the morphism spaces in the following way: For every tetrahedron $t \in \Delta_3$ with vertices $v_0 < v_1 < v_2 < v_3$, denote by $\tau_i$ the face triangle of $t$ where the vertex $v_i$ is left out. Now choose dual bases for the space $\mathcal{D}\left(FY\left(\tau_0\right) \otimes FY\left(\tau_2\right) \otimes FY\left(\tau_1\right) \otimes FY\left(\tau_3\right), I\right)$ and its dual.

Then, using the same convention, an $F$-morphism labelling is a function

$$\iota : \Delta_3 \to \operatorname{mor} \mathcal{D} \tag{5.2.2}$$

where $\iota(t)$ is a basis vector of the space $\mathcal{D}\left(FY\left(\tau_0\right) \otimes FY\left(\tau_2\right) \otimes FY\left(\tau_1\right) \otimes FY\left(\tau_3\right), I\right)$.

**Definition 5.2.** For given labellings $X$ and $\iota$, define as their "amplitude" the evaluation of the labelled link diagram:

$$[X, \iota] := \prod_{s \in \Delta_4} \quad (FX(\tau_i), \iota(t_i)) \tag{5.2.3}$$

Here, the $t_i$ are the faces of $s$ and the $\tau_i$ their faces in turn, in the appropriate order. Whenever the orientation of the $D^3$ of a tetrahedron $t_i$ induced from the total ordering matches the face orientation from the 4-simplex, evaluate the  -quantity with the morphism $\iota(t_i)$ and otherwise with its dual basis vector. Since every tetrahedron is the face of exactly two 4-simplices, for every morphism $\iota(t)$, its dual will appear exactly once in the labelling, so the sum in the following will indeed range over dual bases.

Note that since the 2-handles are labelled with $F\Omega_{\mathcal{C}}$, the  diagram must be labelled with $FX(\tau_i)$.

**Lemma 5.3.** From the normalisation from the multiple vertices, the evaluation of a Kirby diagram $K$ is:

$$\langle K(F) \rangle = d(\Omega_{\mathcal{C}})^{1-|\Delta_4|} \sum_{\substack{\text{labellings} \\ X, \iota}} [X, \iota] \tag{5.2.4}$$

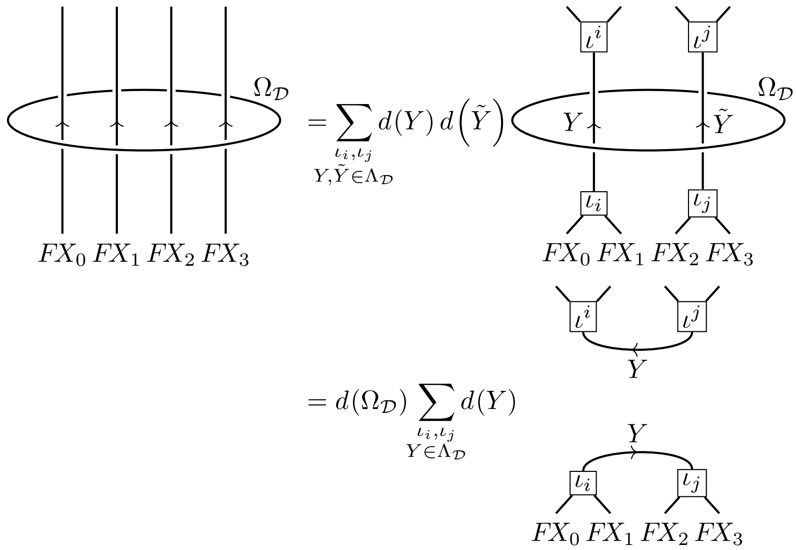
This quantity has to be multiplied by the normalisation $d(\Omega_{\mathcal{C}})^{-h_2+h_1}\, n^{-h_1}$, where $n = d\big((F\Omega_{\mathcal{C}})'\big)$.

**Theorem 5.4.** For $F\colon \mathcal{C} \to \mathcal{D}$ being a pivotal functor satisfying the conditions of 3.5 with $\mathcal{D}$ modular, the generalised dichromatic invariant has a state sum formula:

$$\begin{aligned} I_F(M) &= d(\Omega_{\mathcal{C}})^{1-|\Delta_4|-|\Delta_2|+|\Delta_1|}\, n^{-h_1} \sum_{\substack{\text{labellings} \\ X, \iota}} [X, \iota] \\ &= d(\Omega_{\mathcal{C}})^{1-|\Delta_4|-|\Delta_2|+|\Delta_1|}\, n^{-h_1} \sum_{\substack{\text{labellings} \\ X, \iota}} \prod_{s \in \Delta_4} \quad (FX(\tau_i), \iota(t_i)) \end{aligned} \tag{5.2.5}$$

## 5.3 Trading four-valent for trivalent morphisms

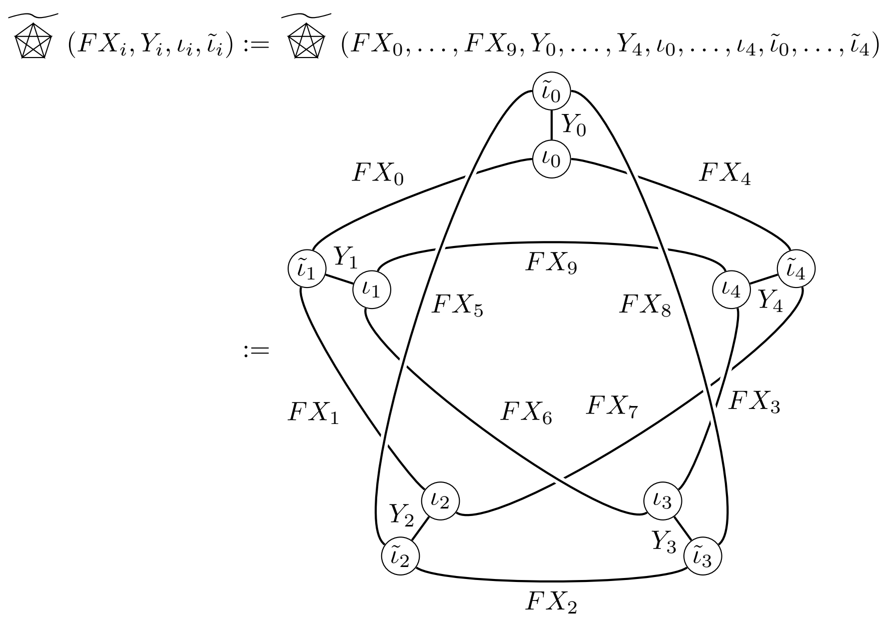
In order to connect to the Crane-Yetter model, the state sum needs to be reformulated slightly. There, the vertices in the  diagram are trivalent, which is more convenient when working with $U_q sl(2)$ tilting modules, but not in a general ribbon fusion category. The four-valent morphisms appeared when applying the lemma 2.39 to the four 2-handles (triangles) going through a 1-handle (tetrahedron) in proposition 4.12. If one inserts two $\Omega_{\mathcal{D}}$s, one can produce two trivalent vertices:




(5.3.1)

For the last step, lemma 2.40 has been used, cancelling a prefactor. Note that the additional objects now range over the simples in $\mathcal{D}$, not $\mathcal{C}$.

The alternative  quantity is then defined as:




(5.3.2)

Again, it has to be specified where an object or its dual is the source or the target of a morphism. Each tetrahedron corresponds to an encirclement. It occurs as the face of two 4-simplices, which are oriented as submanifolds of $M$, and the tetrahedron inherits two opposite orientations from each of them. Orient the encircling (5.3.1) such that the 4-simplex from which the tetrahedron inherits the orientation agreeing with the ordering of the vertices appears on the top.

Object and morphism labellings now have different definitions than in section 5.2:

**Definition 5.5.** An $F$-object labelling of the triangulation $\Delta$ is a pair of functions $(X, Y)$, where

$$X : \Delta_2 \to \Lambda_{\mathcal{C}} \tag{5.3.3}$$

$$Y : \Delta_3 \to \Lambda_{\mathcal{D}} \tag{5.3.4}$$

Choose dual bases for the spaces $\mathcal{D}\left(FX\left(\tau_0\right) \otimes FX\left(\tau_2\right), Y(t)\right)$ and $\mathcal{D}\left(FX\left(\tau_1\right) \otimes FX\left(\tau_3\right), Y(t)\right)$ and their duals.

An $F$-morphism labelling is a pair of functions $(\iota, \tilde{\iota})$

$$\iota, \tilde{\iota} : \Delta_3 \to \operatorname{mor} \mathcal{D} \tag{5.3.5}$$

where $\iota(t)$ is a basis vector of the space $\mathcal{D}\left(FX\left(\tau_0\right) \otimes FX\left(\tau_2\right), Y(t)\right)$ and $\tilde{\iota}(t)$ is a basis vector of $\mathcal{D}\left(FX\left(\tau_1\right) \otimes FX\left(\tau_3\right), Y(t)\right)$.

**Definition 5.6.** For given labellings $(X, Y)$ and $(\iota, \tilde{\iota})$, the amplitude is:

$$\langle (X, Y), (\iota, \tilde{\iota}) \rangle := \prod_{t \in \Delta_3} d(Y(t))\, d(\Omega_{\mathcal{D}}) \prod_{s \in \Delta_4} \widetilde{\bigstar}\, (FX(\tau_i), Y(t_i), \iota(t_i), \tilde{\iota}(t_i)) \tag{5.3.6}$$

**Lemma 5.7.** From the Killing property and the normalisation from the multiple vertices, the evaluation of a special framed link $L$ is:

$$\langle L\left(\Omega_{\mathcal{D}}, F\Omega_{\mathcal{C}}\right) \rangle = d(\Omega_{\mathcal{C}})^{1-|\Delta_4|} \sum_{\substack{\text{labellings} \\ (X,Y),(\iota,\tilde{\iota})}} \langle (X, Y), (\iota, \tilde{\iota}) \rangle \tag{5.3.7}$$

**Theorem 5.8.** The state sum formula can also be written as:

$$\begin{aligned} I_F(M) &= d(\Omega_{\mathcal{C}})^{1-|\Delta_2|+|\Delta_3|-|\Delta_4|}\, d(\Omega_{\mathcal{D}})^{-|\Delta_3|}\, d\big((F\Omega_{\mathcal{C}})'\big)^{-|\Delta_3|} \sum_{\substack{\text{labellings} \\ (X,Y),(\iota,\tilde{\iota})}} \langle (X, Y), (\iota, \tilde{\iota}) \rangle \\ &= d(\Omega_{\mathcal{C}})^{1-\chi(M)+|\Delta_0|-|\Delta_1|}\, d\big((F\Omega_{\mathcal{C}})'\big)^{-|\Delta_3|} \\ &\quad \cdot \sum_{\substack{\text{labellings} \\ (X,Y),(\iota,\tilde{\iota})}} \prod_{t \in \Delta_3} d(Y(t)) \prod_{s \in \Delta_4} \widetilde{\bigstar}\, (FX(\tau_i), Y(t_i), \iota(t_i), \tilde{\iota}(t_i)) \end{aligned} \tag{5.3.8}$$

# 6 Examples

## 6.1 The Crane-Yetter state sum

If $F\colon \mathcal{C} \to \mathcal{D}$ is a full inclusion (Petit's dichromatic invariant, example 3.13) and $\mathcal{D}$ is already modular, the generalised dichromatic invariant simplifies:

**Proposition 6.1.** Let $F\colon \mathcal{C} \hookrightarrow \mathcal{D}$ be a full pivotal inclusion of a spherical fusion category into a modular category.

1. $I_F$ depends only on $\mathcal{C}$, with the inherited ribbon structure.

2. $I_F$ is the Crane-Yetter state sum $CY_{\mathcal{C}}$ for $\mathcal{C}$ up to the Euler characteristic $\chi$:

$$I_F(M) = CY_{\mathcal{C}}(M) \cdot d(\Omega_{\mathcal{C}})^{1-\chi(M)} \tag{6.1.1}$$

*Proof.* 1. Since $\mathcal{D}$ is modular, the simplified definition in proposition 4.12 can be used, with $n = 1$. Object labellings already take values in $\Lambda_{\mathcal{C}}$. Morphism labellings take values in $\mathcal{D}(FX_1 \otimes \cdots \otimes FX_N, I)$, but this is isomorphic to $\mathcal{C}(X_1 \otimes \cdots \otimes X_N, I)$ since $F$ is full. The evaluation of the Kirby diagram can thus be carried out in $\mathcal{C}$ and depends only on data from $\mathcal{C}$ and the ribbon structure inherited from $\mathcal{D}$.

2. In the state sum description, an additional $\Omega_{\mathcal{D}}$ is inserted in (5.3.1) to transform the 4-valent vertex into two 3-valent vertices, introducing additional objects $X$ labelling the tetrahedra. Here, this can be achieved instead by using the insertion lemma 2.32 in $\mathcal{C}$. Thus the labellings of the state sum can be taken to range over $X\colon \Delta_3 \to \Lambda_{\mathcal{C}}$ and $\iota, \tilde{\iota}\colon \Delta_3 \to \operatorname{mor}\mathcal{C}$.

   A direct comparison of the state sum formula (5.3.8) to [CKY97, theorem 3.2] shows the equality to $CY_{\mathcal{C}}$. The version of the insertion lemma 2.32 slightly differs from [CKY97] by inserting $\Omega_{\mathcal{C}} = \bigoplus_X d(X)\, X$ whereas Crane, Yetter and Kauffman insert $\bigoplus_X X$, leading to different dimension factors.

□

*Remark* 6.2. Let $\mathcal{C}$ be a ribbon fusion category with braiding $c_{-,-}$. Then there is a full inclusion of $\mathcal{C}$ into its Drinfeld centre $\mathcal{Z}(\mathcal{C})$ by mapping $X \mapsto (X, c_{X,-})$. So the Crane-Yetter invariant can always be studied as a case of Petit's dichromatic invariant. This is a significant generalisation since the original derivation of the Crane-Yetter state sum from a handlebody picture required $\mathcal{C}$ to be modular, while the version presented here does not.

*Remark* 6.3. Recall that if $\mathcal{D}$ is not modular, but modularisable, then considering the associated state sum model via the modularisation $H$, $H \circ F$ will not always be full and may thus fail to give rise to a case of Petit's dichromatic invariant. However, if both categories are unitary, corollary 4.10 can be used to return to a full inclusion.

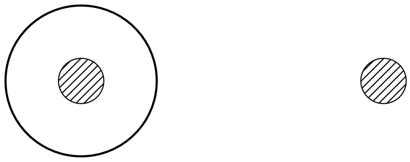
## 6.2 Non-simply-connected manifolds

If $M^4$ is not simply-connected, then the observation in lemma 3.10 (that on simply-connected manifolds, our invariant is not stronger than Euler characteristic and signature) is not applicable any more. And indeed, already the Crane-Yetter invariant is stronger than the Broda invariant on such manifolds, in that it depends at least on the fundamental group. This can be seen in an example.

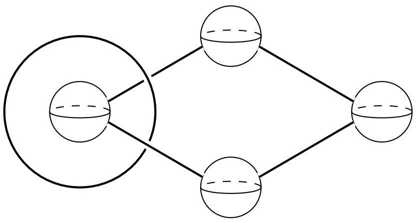
Consider the Crane-Yetter model of a ribbon fusion category $\mathcal{C}$ that is not modular. This is, up to Euler characteristic, the generalised dichromatic invariant $I_F$ for a full inclusion $F$ of $\mathcal{C}$ into a modular category $\mathcal{D}$. The invariant will be evaluated on manifolds of the form $S^1 \times M^3$. Since $S^1 \times M = \partial(D^2 \times M)$, its signature must be 0. The Euler characteristic is also $\chi(S^1 \times M) = \chi(S^1) \cdot \chi(M) = 0$.

Let us study the cases $M = S^3$ and $M = S^1 \times S^2$. The manifold $S^1 \times S^3$ has a handle decomposition with one 1-handle and no 2-handles and its link diagram in Akbulut notation is the dotted unknot. Therefore, $I_F(S^1 \times S^3) = d(\Omega_{\mathcal{C}})$.

Following [GS99, 4.3.1, 4.6.8 and 5.4.2], one arrives at a handle decomposition of $S^1 \times S^1 \times S^2$ starting from a Heegaard diagram of $S^1 \times S^2$ presented in the form of a 2-handle attaching curve on the boundary of a solid torus.

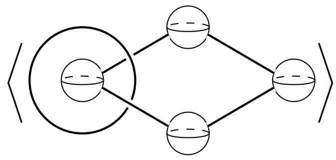
$S^1 \times S^2 =$ 

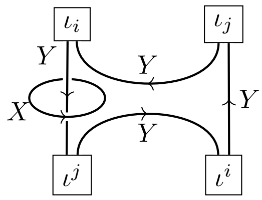
The two disks are the attaching disks of the 1-handle in $\partial D^3 = S^2 = \mathbb{R}^2 \cup \{\infty\}$. The circle is the attaching circle of the 2-handle. Thickening this picture gives a Kirby diagram for $I \times S^1 \times S^2$ and adding a further 1- and 2-handle gives:

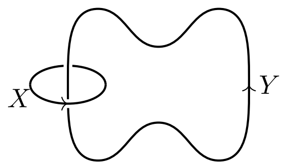
$S^1 \times S^1 \times S^2 =$ 

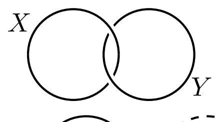
The left and the right 3-ball are the attaching balls of the thickened 1-handle, the front and the back ones come from the additional 1-handle.

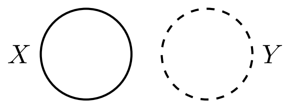
The simplified definition of the invariant from proposition 4.12 is used. Since there are the same number of 2-handles and 1-handles and $n = 1$, the normalisation is 1, and the invariant evaluates to

$I\left(S^1 \times S^1 \times S^2\right) =$ 

$= \sum_{X,Y \in \Lambda_{\mathcal{C}}} d(X)\, d(Y)$ 


$= \sum_{X,Y \in \Lambda_{\mathcal{C}}} d(X)\, d(Y)^{-1}$ 


$= \sum_{X,Y \in \Lambda_{\mathcal{C}}} d(X)\, d(Y)^{-1}$ 


$= \sum_{\substack{X \in \Lambda_{\mathcal{C}} \\ Y \in \Lambda_{\mathcal{C}'}}} d(X)\, d(Y)^{-1}$ 


$$= \sum_{\substack{X \in \Lambda_{\mathcal{C}} \\ Y \in \Lambda_{\mathcal{C}'}}} d(X)^2 \quad = \quad |\Lambda_{\mathcal{C}'}|\, d(\Omega_{\mathcal{C}})\,. \tag{6.2.1}$$

If $\mathcal{C}$ is not modular, that is, if $\Lambda_{\mathcal{C}'}$ has more than one element, $I(S^1 \times S^1 \times S^2) \neq I(S^1 \times S^3)$.

An example of this invariant is the refined Broda invariant described in section 3.5. According to [Rob97], the invariant, with our normalisation, is

$$I_F = 2^{b_1 - 1} d(\Omega_{\mathcal{C}}) \tag{6.2.2}$$

for any manifold of the form $S^1 \times M^3$, $b_1$ being the first $\mathbb{Z}_2$-coefficient Betti number of the four-manifold. The simplest example is for $q = e^{i\pi/4}$, when the simple objects are the half-integer spin representations $\Lambda_{\mathcal{D}} = \left\{0, \frac{1}{2}, 1\right\}$ and $\Lambda_{\mathcal{C}} = \{0, 1\}$. In this example, $\mathcal{C} = \mathcal{C}' \simeq \mathrm{Rep}\,(\mathbb{Z}_2)$ is symmetric monoidal. If one takes a different non-trivial root of unity, $\mathcal{C}$ will not be symmetric monoidal any more, but it still has exactly two transparent objects. Note that our results differ from those reported in [CKY93], where the authors implicitly assumed that $\mathcal{C}$ is modular, which it isn't.

## 6.3 Dijkgraaf-Witten models

The purpose of this section is to show how Dijkgraaf-Witten models are a special case of the Crane-Yetter model, and therefore of Petit's dichromatic invariant. The construction uses the representations of a finite group, for which the same symbol is used for a representation and its underlying vector space. If $\rho_1$ and $\rho_2$ are representations, then the trivial braiding is the map $c_{\rho_1,\rho_2}(x \otimes y) = y \otimes x$.

**Definition 6.4.** Let $F\colon \mathrm{Rep}(G) \hookrightarrow \mathcal{D}$ be a full ribbon inclusion of the representations of a finite group $G$, with the trivial braiding and trivial twist, into a modular category. Then the invariant $I_F$ is called the "Dijkgraaf-Witten invariant" associated to $G$.

*Remark* 6.5. This choice of name will be justified subsequently. Since $F$ is full, $I_F$ only depends on $G$, as argued in section 6.1. Further comments on Dijkgraaf-Witten invariants as Crane-Yetter, or Walker-Wang TQFTs are found in section 7.2.

**Definition 6.6.** The **regular representation** of a finite group $G$ is denoted as $\mathbb{C}[G]$ and defined as follows: The underlying vector space is the free vector space over the set $G$. The action of $G$ is defined on the generators by left multiplication.

It is known that $\mathbb{C}[G] \cong \Omega_{\mathrm{Rep}(G)} \cong \oplus_\rho \rho \otimes \mathbb{C}^{d(\rho)}$ where $\rho$ ranges over the irreducible representations of $G$.

**Definition 6.7.** Every group element $g \in G$ gives rise to a natural transformation of the fibre functor, $\mu(g)_\rho\colon \rho \to \rho$, given by $\mu(g)_\rho(v) = gv$. In fact, $\mu$ is a homomorphism.

The following two lemmas are basic facts of finite group representation theory.

**Lemma 6.8.** For any representation $\rho$, there is a projection on the invariant subspace:

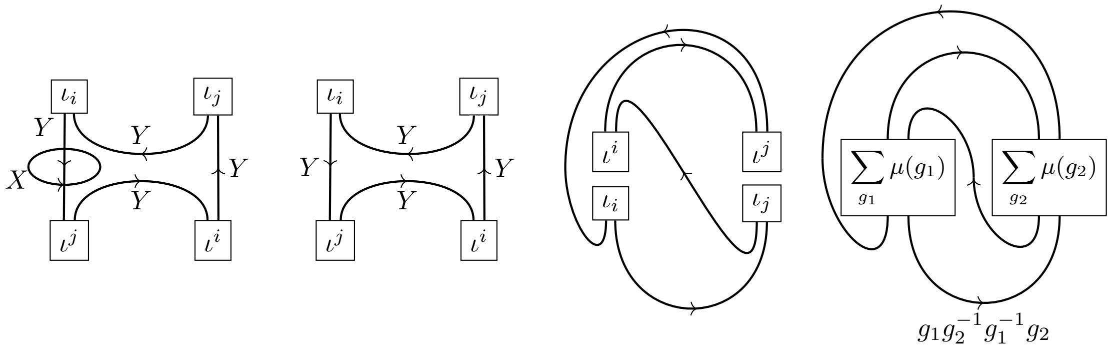
$$\mathrm{inv}_\rho = \sum_i \rho \xrightarrow{\iota_i} I \xrightarrow{\iota^i} \rho \tag{6.3.1}$$

$$= \frac{1}{|G|} \sum_g \mu(g)_\rho \tag{6.3.2}$$

The $\iota_i$ and $\iota^j$ range over bases with $\iota_i \circ \iota^j = \delta_{i,j} 1_I$.

**Lemma 6.9.** The categorical trace over left multiplication on the regular representation, $\mu(g)_{\mathbb{C}[G]}$, is proportional to the delta function:

$$\mathrm{tr}\left(\mu(g)_{\mathbb{C}[G]}\right) = |G| \delta(g) \tag{6.3.3}$$




Evaluation of handle picture of a non-simply connected manifold

The 2-handles not attached to 1-handles are cancelled

Rearrange 1-handles to recognise projection morphisms

Identify 1-handles with generators of the fundamental group and trace with relation words

Figure 1: Evaluating Dijkgraaf-Witten theory

**Definition 6.10.** For a finite group $G$, a flat $G$-connection on a topological space $M$ is a homomorphism $\pi_1(M) \to G$.

*Remark* 6.11. Only connections on 4-manifolds will be considered here. Recall from 2.3.3 that the generators of the fundamental group $\pi_1(M)$ are given by the 1-handles, while each 2-handle is a relation word. Then a homomorphism $\pi_1(X) \to G$ is a choice of a group element for each 1-handle such that for every 2-handle, the group elements according to its relation word compose to the trivial element.

The following result shows that this invariant depends only on $\pi_1(M)$.

**Theorem 6.12.** *Let $F\colon \operatorname{Rep}(G) \hookrightarrow \mathcal{D}$ be a full ribbon inclusion of the representations of a finite group $G$ (with the trivial braiding) into a modular category. Then $I_F(M)$ is the number of flat $G$-connections on $M$.*

*Proof.* The proof is graphical. Since $\mathcal{D}$ is modular, the simplified definition of the invariant from proposition 4.12 can be used. Since $F$ is full, the invariant can be calculated using objects and morphisms from $\mathcal{C}$, as in proposition 6.1. The morphism $K(F)$ can be manipulated using the coherence axioms of ribbon categories as isotopies of the link in the plane. An example is given in figure 1, though one should bear in mind that in general there may be more than two 2-handle attaching curves passing along each 1-handle. There may also be crossings that cannot be removed by an isotopy.

Consider the 2-handles in the link picture that are not linked to a 1-handle. Since $\operatorname{Rep}(G)$ is symmetric with trivial twist and $F$ is ribbon, all knots and framings on them can be undone, and then the morphism can be isotoped away. These 2-handles can be cancelled with the normalisation, arriving at a link picture that, while evaluating to the same invariant, has only 2-handles which start or end in morphisms coming from 1-handles.

The morphisms on 1-handles are lined up horizontally and, after an isotopy, recognised as the projection morphisms $\operatorname{inv} = \frac{1}{|G|} \sum_g \mu(g)$ defined in lemma 6.8. All of the 1-handles then give a morphism $\frac{1}{|G|^{h_1}} \sum_{g_1} \mu(g_1) \otimes \sum_{g_2} \mu(g_2) \otimes \cdots \otimes \sum_{g_{h_1}} \mu(g_{h_1})$, which are trace over with the 2-handles. The factor $\frac{1}{|G|^{h_1}}$ is cancelled by the normalisation as well since $\Omega_{\mathcal{C}} = |G|$.

To perform the trace for each 2-handle, consider lemma 6.9. If the relation word for the 2-handle $k$ is denoted by $r_1 r_2 \dots r_{m_k}$, the trace for $k$ is $\delta\left(g_{r_1} g_{r_2} \cdots g_{r_{m_k}}\right)$. Again the remaining normalisation is cancelled with the factor $|G|$. After tracing out with all 2-handles, the invariant is then

$$\begin{aligned} I_F(M) &= \sum_{g_1 \in G} \sum_{g_2 \in G} \cdots \sum_{g_{h_1} \in G} \prod_{\text{2-handles } k} \delta\left(g_{r_1} g_{r_2} \cdots g_{r_{m_k}}\right) \\ &= |\{\phi \colon \pi_1(M) \to G\}| \end{aligned} \tag{6.3.4}$$

using remark 6.11. □

This result shows that $I_F$ is the partition function of a Dijkgraaf-Witten model, described for example in [Yet92]. In the more common normalisation in the literature, one would divide $I_F$ by $|G| = d(\Omega_{\mathcal{C}})$, though.

*Remark* 6.13. One would expect a four-dimensional Dijkgraaf-Witten model to depend not only on a finite group $G$, but also on a 4-cocycle on $G$. The cocycle in the present model is trivial, though. A natural way for a 4-cocycle to arise is as pentagonator in a tricategory. But braided categories are a special case of a tricategory with one 1-morphism, and these have a trivial pentagonator, see e.g. [CG07]. Hence, there seems little hope to introduce the data of a 4-cocycle into the representation category of $G$. The model would have to be generalised to fully weak monoidal bicategories, for example, following e.g. [Mac99].

*Remark* 6.14. Due to the Doplicher-Roberts reconstruction (see [Dri+10, paragraph 2.12] for a categorical approach), it is known that symmetric fusion categories with trivial twist are essentially representations of finite supergroups. If the dimensions of all objects are required to be positive, the supergroup is in fact a group. So the case studied here is not much more restrictive than demanding that $\mathcal{C}$ be a symmetric fusion category.

## 6.4 Invariants from group homomorphisms

It is natural to consider generalising the Dijkgraaf-Witten examples by replacing the group $G$ with a homomorphism $\phi\colon P \to G$. Any homomorphism can be factored into a surjective homomorphism followed by an inclusion, as $P \to \operatorname{Im}\phi \to G$. Taking the categories of unitary finite-dimensional representations leads to a functor

$$\phi^* \colon \operatorname{Rep}(G) \to \operatorname{Rep}(P)$$

given by composition with $\phi$. It factors into functors $A\colon \operatorname{Rep}(G) \to \operatorname{Rep}(\operatorname{Im}\phi)$ followed by $B\colon \operatorname{Rep}(\operatorname{Im}\phi) \to \operatorname{Rep}(P)$. The first functor $A$ is a restriction functor, which is a dominant functor. This follows from the fact that for any $\operatorname{Im}\phi$-representation $\rho$, $\rho \subset A \operatorname{Ind}\rho$, where Ind is the induction functor to $P$-representations. The second functor $B$ is a full inclusion.

### 6.4.1 Trivial braiding

The first case to consider is when $\operatorname{Rep}(P)$ is augmented with the trivial braiding and trivial twist to make it a ribbon category, as in the Dijkgraaf-Witten invariant. Let $F\colon \operatorname{Rep}(P) \hookrightarrow \mathcal{D}$ be a full ribbon inclusion of $\operatorname{Rep}(P)$ with this ribbon structure into a modular category.

Then the invariant $I_{F\circ\phi^*}$ generalises the Dijkgraaf-Witten invariant in principle but its evaluation is the same as a Dijkgraaf-Witten invariant. Indeed $F \circ \phi^* = F \circ B \circ A$. But $A$ is dominant unitary and can be cancelled using proposition 4.1, while $F \circ B$ is a full ribbon inclusion of $\operatorname{Rep}(\operatorname{Im}\phi)$ in $\mathcal{D}$, and so defines a Dijkgraaf-Witten invariant.

Despite the fact that the invariant is not new, the construction is still interesting because it may be a starting point for physical models. Just as in proposition 6.1, the invariant can be calculated in the category $\mathrm{Rep}(P)$. The object labels are simple objects $X_i \in \mathrm{Rep}(G)$ and the morphism labels are a basis in $\mathrm{Rep}(P)\,(\phi^* X_1 \otimes \ldots \otimes \phi^* X_N, I)$, or its dual space. The invariant is evaluated using the representation $p \mapsto \phi^* \mu_{\mathbb{C}[G]}(p) = \mu_{\mathbb{C}[G]}(\phi(p))$ with trace

$$\operatorname{tr} \mu_{\mathbb{C}[G]}\left(\phi(p)\right) = |G| \delta\left(\phi(p)\right)$$

using the delta-function in $G$. The projection morphisms are

$$\frac{1}{|P|} \sum_p \mu\left(\phi(p)\right).$$

Since the functor $F$ is a full inclusion, the multiplicity $n$ is just the multiplicity of $I$ in $\phi^* \mathbb{C}[G]$. This can be calculated as $n = |G| / |\operatorname{Im} \phi|$. The formula for the invariant is thus

$$I_{F \circ \phi^*}(M) = \frac{1}{|\operatorname{Ker} \phi|^{h_1}} \sum_{p_1 \in P} \sum_{p_2 \in P} \cdots \sum_{p_{h_1} \in P} \prod_{2\text{-handles } k} \delta\left(\phi(p_{r_1} p_{r_2} \cdots p_{r_{m_k}})\right) \tag{6.4.1}$$

Immediately, one can see that one can replace the $\delta$-function in $G$ by the one in $\operatorname{Im} \phi$ without changing the value of the invariant. Also each group element $\phi(p)$ appears exactly $|\operatorname{Ker} \phi|$ times, cancelling the normalisation. Thus one sees explicitly that the manifold invariant is the Dijkgraaf-Witten invariant of the subgroup $\operatorname{Im} \phi \subset G$.

### 6.4.2 Non-trivial braiding

A different construction from a group homomorphism is to consider cases where $\mathrm{Rep}(P)$ is augmented with a non-trivial braiding so that it becomes a ribbon category. Then one can consider the invariant $I_{\phi^*}$ directly, without needing the functor to a modular category. Of course this also works with the trivial braiding but then we can apply proposition 4.1 with the fibre functor to vector spaces and the invariant is equal to 1.

*Example* 6.15. If $\phi \colon P \to G$ is injective, then $I_{\phi^*} = I_{1_{\mathrm{Rep}(P)}}$, which is a Broda invariant for the category $\mathrm{Rep}\, P$ and depends only on the Euler number and signature of the four-manifold.

*Example* 6.16. If $\phi \colon P \to G$ is surjective, then $I_{\phi^*}$ is a Petit dichromatic invariant.

The simplest examples are where $P = \mathbb{Z}_n$ is the cyclic group of order $n$ with the anyonic braiding [Maj00, example 2.1.6] and the pivotal structure from Vect. The irreducible representations are one-dimensional and also labelled by $\mathbb{Z}_n$. The braiding on two irreducibles $k, k'$ is

$$x \otimes y \mapsto e^{\frac{2\pi i}{n} k k'} y \otimes x$$

and so the transparent objects are $k = 0$, and also $k = n/2$ if $n$ is even. In the case that $n$ is odd, $\mathrm{Rep}(\mathbb{Z}_n)$ is modular and so the invariant of example 6.16 only depends on $\mathrm{Rep}(G)$ with its induced ribbon structure. It is a Crane-Yetter invariant.

There are many more possible braidings [Dav97] and it seems an interesting project to explore the corresponding constructions of the invariant and Crane-Yetter models, which is left for future work.

# 7 Relations to TQFTs and physical models

This discussion section is written in a more informal style. The invariants defined in this paper are related to various physical models. It is not just the value of the invariant that is important but also its construction in terms of data on simplexes or handles. This is because in a physical model one is interested in features that are localised to lower-dimensional subsets, such as boundaries, corners or defects associated to embedded graphs, surfaces or other strata. In some cases it is possible to identify this data as the discrete version of a field in quantum field theory.

## 7.1 TQFTs from state sum models

Whenever there is a state sum formula for $I_F$, that is, when $\mathcal{D}$ is modular, it is possible to cast it in the form of a Topological Quantum Field Theory (TQFT) $\mathcal{Z}$, following a standard recipe [TV92].

- For a boundary manifold $M^3$ with a given triangulation $\Delta$, define the set of labellings $L(M,\Delta)$ exactly like for the state sum model in definition 5.1. Then define the free complex vector space $Y(M,\Delta) := \mathbb{C}[L(M,\Delta)]$.
- For a cobordism $\Sigma^4\colon M_1 \to M_2$ with triangulation $\Delta$, the amplitude $\langle l_1 | \, U(\Sigma,\Delta) \, | l_2 \rangle$ is defined for the basis vectors coming from $l_{1,2} \in L\left(M_{1,2}, \Delta|_{1,2}\right)$ via the state sum: Sum over all labellings of $\Sigma$ that have $l_1$ and $l_2$ as boundary conditions. This gives a linear map $U(\Sigma,\Delta)\colon L\left(M_1,\Delta|_1\right) \to L\left(M_2,\Delta|_2\right)$. It is independent of the triangulation in the interior.
- The TQFT on an object $M^3$ is defined as the image of $U(I\times M)$. These spaces for different triangulations can be identified in a coherent way, again using cylinders, giving a vector space that is independent of the triangulation of $M$.
- The TQFT on morphisms $\Sigma$ is defined by the restriction of $U$ to the aforementioned spaces. Since a cylinder can always be glued to a cobordism without changing its isomorphism class, this is well-defined.

## 7.2 Walker-Wang models and the toric code

It is believed that Walker-Wang TQFTs [WW12] are the Hamiltonian formulation of Crane-Yetter TQFTs. Therefore, it seems natural to conjecture the following: For modular $\mathcal{D}$, Petit's dichromatic invariant for a full inclusion $F\colon \mathcal{C} \hookrightarrow \mathcal{D}$ extends to a Topological Quantum Field Theory $\mathcal{Z}$ of Walker-Wang type. More precisely, for a closed cobordism $\Sigma^4$,

$$\mathcal{Z}(\Sigma) = \frac{I_F(\Sigma)}{d(\Omega_{\mathcal{C}})^{1-\chi(\Sigma)}} \tag{7.2.1}$$

The denominator $d(\Omega_{\mathcal{C}})^{1-\chi(\Sigma)}$ is provided by comparison to the Crane-Yetter state sum (6.1.1).

Furthermore, it seems a good conjecture that $\mathcal{Z}$ on objects, that is, the state spaces of the TQFT, are skein (or "string net") spaces labelled by the symmetric centre $\mathcal{C}'$. This would imply that the dimensions of these state spaces for boundary manifolds $M^3$ can be calculated:

$$\begin{aligned}\dim \mathcal{Z}(M) &= \operatorname{tr} \mathbb{1}_{\mathcal{Z}(M)} \\ &= \mathcal{Z}\left(S^1 \times M\right) \\ &= \frac{I_F\left(S^1 \times M\right)}{d(\Omega_{\mathcal{C}})}\end{aligned} \tag{7.2.2}$$

Non-trivial values of the invariant for manifolds of the form $S^1 \times M^3$ can then be interpreted as dimensions of state spaces of the corresponding TQFT. Comparing with section 6.2 shows that these dimensions can indeed be greater than 1, as in the example of Broda's refined invariant.

As an example, for $M = S^1 \times S^2$, one arrives at $\dim \mathcal{Z}(S^1 \times S^2) = |\Lambda_{\mathcal{C}'}|$. This result is in excellent agreement with the analysis of Walker-Wang ground state degeneracies in [CBS13]. The state space of a TQFT corresponds to the space of ground states of the Hamiltonian.

A known special case is again the refined Broda invariant, mentioned in 6.2. Choosing the deformation parameter $q = \mathrm{e}^{\frac{\mathrm{i}\pi}{4}}$ results in $\mathcal{C} \simeq \mathbb{Z}_2$ and the resulting model is the 3+1-dimensional generalisation of the toric code [LW05] studied in [CBS13].

If $\mathcal{C} \simeq \mathrm{Rep}(G)$ for $G$ a finite group, the dimensions can be calculated explicitly, recalling section 6.3:

$$\begin{aligned}
\frac{I_F\left(S^1 \times M\right)}{d(\Omega_{\mathcal{C}})} &= \frac{\left|\left\{\phi\colon \pi_1\left(S^1 \times M\right) \to G\right\}\right|}{|G|} \\
&= \frac{\left|\left\{\phi\colon \mathbb{Z} \times \pi_1(M) \to G\right\}\right|}{|G|} \\
&= \frac{\left|\left\{\left(\phi\colon \pi_1(M) \to G, g \in G\right) \middle| \phi = g\phi g^{-1}\right\}\right|}{|G|} \\
\text{(By Burnside's lemma)} \qquad &= \left|\left\{\phi\colon \pi_1(M) \to G\right\} / \sim\right| \qquad \text{where } \phi \sim g\phi g^{-1} \qquad (7.2.3)
\end{aligned}$$

The state spaces are thus spanned by conjugacy classes of connections on the boundary manifolds. This suggests that $I_F$ indeed extends as a Dijkgraaf-Witten TQFT.

## 7.3 Quantum gravity models

General relativity can be formulated in terms of connections and so it is natural to construct state sum models, or more generally quantum invariants of manifolds, that are modelled on connections. Usually the groups are Lie groups but these do not lead to fusion categories since the number of irreducibles is not finite. As a toy model therefore one can replace the Lie groups by finite groups to get an easy comparison with some of the invariants constructed above. A more sophisticated resolution of this problem is to use instead representations of quantum groups at a root of unity, which are indeed fusion categories. Finite groups are discussed here first and then some comments on the obstruction to using quantum groups in a similar way are made below.

Cartan connections can be thought of as principal $G$-connections that allow only gauge transformations of a subgroup $P \hookrightarrow G$. One of the motivations for the development of the generalised dichromatic invariant was the hope of arriving at a state sum model that could be interpreted as quantum Cartan geometry. Since there are formulations of general relativity in terms of Cartan geometry (see e.g. [Wis10]), this would give an interesting new approach to quantum gravity. However our constructions in 6.4 based on an inclusion $P \hookrightarrow G$ do not appear to lead to interesting new models.

A closely related construction is teleparallel gravity. This is based on a surjective homomorphism $P \to G$ with kernel $N$. According to Baez and Wise [BW12, theorem 32] the data for teleparallel gravity is a flat $G$-connection and a 1-form with values in the Lie algebra of $N$. For them, $P$ is the Poincaré group and $N$ the translation subgroup, but here the groups are allowed to be more general.

A flat $G$ connection is easily described as an assignment of an element $g \in G$ to each 1-handle with a relation on each 2-handle, as in the Dijkgraaf-Witten model. The discrete analogue of the 1-form is the assignment of an element $n \in N$ to each 1-handle, with no relations on this data. For finite groups, this is exactly the data that is summed over in (6.4.1), the invariant associated to the homomorphism $\phi\colon P \to G$ that has kernel $N$. Two elements $p, p' \in P$ such that $\phi(p) = \phi(p')$ differ by an element $p^{-1}p' \in N$. This is the discrete analogue of the fact that the difference of two connection forms on a manifold is a 1-form. Thus the construction in (6.4.1) is a plausible finite group analogue of a sum over configurations of teleparallel gravity.

### 7.3.1 Quantum groups

Classical geometry works with Lie groups, which have an infinite number of irreducible representations. One hope would be to use quantum groups at a root of unity as a regularisation. However, few Lie group homomorphisms carry over to quantum groups. There are many examples of subgroups of Lie groups, but fewer sub-quantum groups of quantum groups are known. This is because most Lie group homomorphisms do not preserve the root system of the Lie algebras and thus neither the deformation. And even for Hopf algebra homomorphisms, the restriction functor is not necessarily pivotal:

*Example* 7.1. As an example of a restriction functor that isn't pivotal, consider the category of tilting modules of $U_q sl(2)$ at an $n$-th root of unity. Its simple objects are spins $j \in \{0, \frac{1}{2}, \dots\}$. Recall that $\mathbb{C}[\mathbb{Z}_n]$ is a sub-Hopf algebra of $U_q sl(2)$. Recalling that $S^2 = SU(2)/U(1)$, one would hope that this Hopf algebra inclusion serves as Cartan geometry with a quantum 2-sphere. The irreducible representations of $\mathbb{C}[\mathbb{Z}_n]$ are Fourier modes $\dots, -1, 0, 1, \dots$. Consider the restriction functor of representations, Res. It is obviously monoidal. Then $\operatorname{Res}\left(\frac{1}{2}\right) = -1 \oplus 1$. Both summands are invertible and thus have dimensions 1, whereas the quantum dimension of $\frac{1}{2}$ is generally not even an integer. Thus Res does not preserve quantum dimensions and can't be pivotal.

The crucial problem here is that the inclusion does not map the spherical element of $\mathbb{C}[\mathbb{Z}_n]$, which is 1, onto the spherical element of $U_q sl(2)$. A quantum group homomorphism of spherical quantum groups that preserves the spherical elements always gives rise to a pivotal functor on the representation categories [BMS12, example 8.5]. However, no such homomorphism that gives rise to an invariant that is not a combination of the previously studied cases is known to the authors.

### 7.3.2 Spin foam models

Spin foam models are state sum models for quantum gravity constructed using representations of a quantum group, originally the "spins" of $U_q sl(2)$, hence the name. Starting with a Crane-Yetter state sum, a popular strategy in spin foam models is to impose constraints on the labels on the triangles and tetrahedra to mimick approaches to gravity as a constrained $BF$-theory [Bae00]. The unconstrained theory corresponds to the Crane-Yetter state sum, and different quantisation strategies of the classical constraints lead to different constraints, like in the Barrett-Crane [BC98] or the EPRL-model [Eng+08]. However, in these models the constraints on objects and morphisms typically spoil the monoidal product and so are not examples of the constructions in this paper. An interesting question is whether it is possible to construct spin foam models of the type considered here, for example a spin foam model for teleparallel gravity. Such a model would involve studying the question of whether there are interesting quantum group analogues of a surjective homomorphism of groups.

## 7.4 Nonunitary theories

There are two possibilities to arrive at a theory which might be more general than the Crane-Yetter model. The first is to drop the assumption of the target category being modularisable;

however this is a mild assumption which only specialises from supergroups to groups. Alternatively, on dropping the assumption that the categories are unitary, lemma 4.3 is not applicable any more. To the knowledge of the authors, it is not known whether for a dominant pivotal functor will always satisfy $F\Omega_{\mathcal{C}} = n \cdot \Omega_{\mathcal{D}}$, so a counterexample might lead to an invariant that can't be reduced to a Crane-Yetter model.

### 7.5 Extended TQFTs

It is a common assumption that the Crane-Yetter model for modular $\mathcal{C}$ is an invertible four-dimensional extended TQFT. According to the cobordism hypothesis, it should correspond to an invertible (and therefore fully dualisable) object in a 4-category. The 4-category in question has as objects braided monoidal categories, as 1-morphisms monoidal bimodule categories (with an isomorphism between left and right action compatible with the braiding), as 2-morphism linear bimodule categories, and furthermore bimodule functors and natural transformations.

A ribbon fusion category $\mathcal{C}$ acting on a spherical fusion category $\mathcal{M}$ from left and right should be an example for a fully dualisable, potentially noninvertible object. The object is $\mathcal{C}$ itself, while its dualisation data on the 1-morphism level is the bimodule data of $\mathcal{M}$. Being a fusion category, $\mathcal{M}$ is a bimodule over itself, giving the 2-morphism level of dualisation. The higher levels of dualisation should correspond to finite semisimplicity.

As has been suggested recently [HPT15, section 3.2], a good notion of monoidal bimodule over a braided category is a braided central functor from $\mathcal{C}$ to $\mathcal{M}$, i.e. a braided functor $F: \mathcal{C} \to \mathcal{Z}(\mathcal{M})$. One would expect that the extended TQFT corresponding to such a bimodule is an extension of our (properly normalised) invariant for $F$, whenever it is also pivotal. And indeed, the inclusion $\mathcal{C} \to \mathcal{Z}(\mathcal{C})$ yields the Crane-Yetter model for $\mathcal{C}$.

## 8 Outlook

The generalised dichromatic invariant is a very versatile invariant in that it contains many previously studied theories as special cases. Here is an overview which functors give rise to several special cases, up to a factor of the Euler characteristic:

| Model | Pivotal functor $F$ | Discussion |
|---|---|---|
| $U_q sl(2)$-Crane-Yetter state sum, Broda invariant | $1_{\mathcal{C}}: \mathcal{C} \to \mathcal{C}$ for $\mathcal{C}$ the tilting modules (spins) of $U_q sl(2)$ | Example 3.12 |
| Refined Broda invariant with $q = e^{i\pi/4}$, toric code | Canonical inclusion $\mathcal{C} \hookrightarrow \mathcal{D}$ for $\mathcal{C} \simeq \operatorname{Rep} \mathbb{Z}_2$ generated by spins $\{0, 1\}$ and $\mathcal{D}$ all spins $\left\{0, \frac{1}{2}, 2\right\}$ | Sections 6.2, 3.5 and 7.2 |
| Refined Broda invariant, Crane-Yetter model for integer spins | Canonical inclusion $\mathcal{C} \hookrightarrow \mathcal{D}$ for $\mathcal{C}$ integer spins and $\mathcal{D}$ all spins | Sections 6.2 and 3.5 |
| Dijkgraaf-Witten TQFT for a finite group $G$ | Any full inclusion of $\operatorname{Rep}(G)$ into a modular category, e.g. canonical inclusion $\operatorname{Rep}(G) \hookrightarrow \mathcal{Z}(\operatorname{Rep}(G))$ | Sections 6.3 and 7.2 |
| General Crane-Yetter state sum, Walker-Wang TQFT for $\mathcal{C}$ any ribbon fusion category | Any full inclusion of $\mathcal{C}$ into a modular category, e.g. canonical inclusion $\mathcal{C} \hookrightarrow \mathcal{Z}(\mathcal{C})$ | Sections 6.1 and 7.2 |
| Petit's dichromatic invariant | Any full inclusion $F: \mathcal{C} \hookrightarrow \mathcal{D}$ for $\mathcal{C}$ and $\mathcal{D}$ ribbon fusion categories | Example 3.13 |
| "Generalised dichromatic state sum models" | Any functor into a modular category | Section 5.2 |

The generalised dichromatic invariant is at least as strong as the Crane-Yetter invariant, which is stronger than Euler characteristic and signature, although it is not known how strong exactly. Imposing the additional constraints that the pivotal functor is unitary and the target category is modularisable, then the generalised dichromatic invariant is exactly as strong as $CY$. In this situation, an upper bound for the strength of the state sum formula is probably given in [Fre+05]: unitary four-dimensional TQFTs cannot distinguish homotopy equivalent simply-connected manifolds, or in general, s-cobordant manifolds. It remains to be demonstrated whether it is possible to construct a stronger, nonunitary TQFT with the present framework.

It is indicated in the literature [WW12] that the Walker-Wang model – and therefore also $CY$ – for an arbitrary ribbon fusion category should factor into $CY$ of its modularisation and its symmetric centre. The former reduces to the signature and the latter has been shown here to depend only on the fundamental group in the case of the symmetric centre being just the representations of a finite group. Supergroups have not been treated here, but one would not expect it to differ much, except possibly a sensitivity to spin structures in the same manner as in the refined Broda invariant (section 3.5).

The question whether the general case of the framework presented here is stronger than the mentioned special cases still remains open. Nevertheless, motivated from solid state physics and TQFTs it would still be interesting to study how defects behave in the new models.

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