Lecture 3: Matrix geometries

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The Project

Replace space-time manifold $\mathcal{A} = C^\infty(M)$ with

$$\mathcal{A} = \text{ matrices}$$

- Fix $s, \mathcal{H}, \mathcal{A}, \Gamma, J$
- $\mathcal{H}$ finite-dimensional
- $\mathcal{G} = \{D\}$ is the space of 'matrix geometries'.
- $\mathcal{G}$ is a f.d. vector space

Note: some geometries are singular (e.g., $D = 0$). Additional properties required to characterise non-singular ones.

Quantum geometry

Pick a bosonic action $S_1(D)$.

Observable $f(D)$.

Define

$$\langle f \rangle = \int_{\mathcal{G}} e^{iS_1(D)} f(D) \,\mathrm{dvol}_{\mathcal{G}}$$

Quantum physics

Pick a bosonic action $S_2(D)$.

Observable $F(D, \psi, \overline{\psi})$, $\quad \psi \in \mathcal{H}$.

Define

$$\langle F \rangle \quad = \quad \int_{\mathcal{G}} \int_{\mathcal{H}} e^{i(S_2(D) + \langle \psi, D\psi \rangle)} \, F(D, \psi, \overline{\psi}) \, \mathrm{d}\psi \, \mathrm{d}\overline{\psi} \, \mathrm{dvol}_{\mathcal{G}}$$

- Integration over $\mathcal{H}$ is Berezin integration
- No additional cutoff required (size of matrices)

Fuzzy spaces

The simplest examples have $\mathcal{A} = M_n(\mathbb{C})$ (over $\mathbb{R}$).

- a^* is Hermitian conjugate
- Pick $p, q \in \{0, 1, 2, \ldots\}$. Then $s = q - p$.
- Define γ^a, signature $p+ \quad q-$, acting in $V \quad (\cdot, \cdot)$.
- $\mathcal{H} = V \otimes M_n(\mathbb{C})$
- $\langle v \otimes m, v' \otimes m' \rangle = (v, v') \, \mathrm{Tr}(m^* m')$
- Action $a(v \otimes m) = v \otimes am$
- Chirality $\Gamma(v \otimes m) = (\gamma v) \otimes m$
- Real structure $J(v \otimes m) = (Cv) \otimes m^*$

Bimodule action

Compute the right action

$$\begin{aligned} Jb^* J^{-1}(v \otimes m) &= Jb^* \Big((C^{-1} v) \otimes m^* \Big) \\ &= J\Big((C^{-1} v) \otimes b^* m^* \Big) \\ &= (CC^{-1} v) \otimes (b^* m^*)^* \\ &= v \otimes mb \end{aligned}$$

Dirac operator

even products of γ^a

odd

$\Omega = \Omega^+ \oplus \Omega^-$: Clifford algebra generated by γ^a

Pick a basis $\{\alpha^j, \beta^k\}$ of Ω^- such that the α^j are anti-Hermitian and the β^k are Hermitian.

$$D(v \otimes m) = \sum_j \alpha^j v \otimes [L^j, m] + \sum_k \beta^k v \otimes \{H^k, m\}$$

$L^j \in \mathcal{A}$ are arbitrary anti-Hermitian elements.

$H^k \in \mathcal{A}$ are arbitrary Hermitian elements.

Examples, $p + q \leq 2$

- Type (0,0) = (p,q)

$$D = 0$$

- Type (1,0)

$$D = \{H, \cdot\}$$

- Type (0,1)

$$D = i[L, \cdot]$$

- Type (2,0)

$$D = \gamma^1 \otimes \{H^1, \cdot\} + \gamma^2 \otimes \{H^2, \cdot\}$$

- Type (1,1)

$$D = \gamma^1 \otimes \{H, \cdot\} + \gamma^2 \otimes [L, \cdot]$$

- Type (0,2)

$$D = \gamma^1 \otimes [L^1, \cdot] + \gamma^2 \otimes [L^2, \cdot]$$

Fuzzy spheres

The fuzzy 2-sphere with a Laplace operator was defined in [Madore 1992 Fuzzy sphere].

The fuzzy 2-sphere with a ‘modified’ Dirac operator d_{GP} was defined by [Grosse, Presnajder 1995] but with no chirality operator.

For an N-sphere, take $p = 1$, $q = N + 1$, so $s = q - p = N$. New

Pick a representation of $\mathrm{SO}(N+1)$ on $\mathbb{C}^n$, Lie algebra generators L_{jk}

$$D = \sum_{j<k=1}^{N+1} \gamma^0 \gamma^j \gamma^k \otimes [L_{jk}, \cdot] + \frac{N}{2}\gamma^0$$

Hermitian

Invariant under Spin$(N+1)$, generators $\gamma^j \gamma^k$ $j,k=1,\dots N$

Comparison with commutative geometry

N+1

- Split into irreps of $\mathrm{SO}(N)$: $\mathcal{H} = \oplus_j \mathcal{H}_j$
- Eigenvalue of D on $\mathcal{H}_j$ same as for classical Dirac operator on commutative S^N
- But different irreps can appear in NC case.

Examples with $\mathbb{C}^n$ irrep of $\mathrm{SO}(N+1)$

Fuzzy S^1 $\quad \mathcal{H} = \mathbb{C}^2$

$$D = \frac{1}{2}\gamma^0, \qquad \text{eigenvalues} \quad \pm\frac{1}{2}$$

classical $\pm\left(\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\right)$

Fuzzy S^2 $\quad \mathcal{H} = \mathbb{C}^4 \otimes M_n(\mathbb{C})$

$q=3,\ p=1 \qquad s=q-p=2$

$$D = \begin{pmatrix} 0 & d_{GP} \\ d_{GP} & 0 \end{pmatrix} \qquad \text{eigenvalues} \quad \pm\{1, 2, \ldots, n\}$$

diag $\begin{pmatrix} d_{GP} & 0 \\ 0 & -d_{GP} \end{pmatrix}$ classical $\pm\{1, 2, \ldots\}$

$$\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\begin{pmatrix} 0 & \cdot \\ \cdot & 0 \end{pmatrix}$

$\pm\{1, 2, \ldots n-1\}$ mult 2

Fermion doubling (except $\pm n$).